

# 9 Relations

## 9.1 Relations and Their Properties

**Def. 1** Let  $A$  and  $B$  be two set. Then a **binary relation**  $R$  from  $A$  to  $B$  is subset of  $A \times B$ .

$$R \subseteq A \times B.$$

$A$ : **domain** of the relation  $R$ .  $B$ : **range(codomain)** of the relation  $R$ .

Let  $a \in A$ ,  $b \in B$ , Then  $(a, b) \in R$  or  $(a, b) \notin R$ .

If  $(a, b) \in R$ , we also write  $a R b$  and we say  $a$  is **related to**  $b$  by  $R$ .

If  $(a, b) \notin R$ , we also write  $a \not R b$  and  $a$  is **not related to**  $b$  by  $R$ .

Two **aspects** of Relation

$R \subseteq A \times B$  relation  $R$  is a **set of pairs**

$$(a, b) \in R$$

$R: A \times B \rightarrow \{T, F\}$  relation  $R$  is a infix **boolean binary operation**

$$a R b$$

Ex.  $(3, 3) \in =$  and  $(3, 4) \notin =$ ; or  $3 = 3$  and  $3 \neq 4$ .

**Function is a Relation but relation is not a function!**

$R \subseteq A \times B$  vs  $f: A \rightarrow B \quad \forall a \in A: \exists_1 f(a) = b \in B.$

If  $f(a) = b$ , then we can write  $(a, b) \in f$  or  $a f b$ .

$\therefore$  Function is a (**special** kind of) relation.

Relation is **not** a function

i) If  $\exists a \in A . \exists . a R b_1$  and  $a R b_2, (b_1 \neq b_2)$ ,  $R$  is **not** a function.

function must have an **unique** related image.

ii) If  $\exists a \in A$  and  $\nexists b \in B . \exists . a R b$ ,  $R$  is **not** a function.

**all** the elements in the domain must have its related image

Let  $R \subseteq A \times B$ . Then

We write  $R(a) = \{b_1, b_2, \dots, b_n\} \in R$ , if  $1 \leq \forall i \leq n: (a, b_i) \in R$ .

**A function is a special kind(subclass) of a relation**

**Three faces of the relation  $R$  from  $A$  to  $B$ .**

**i)  $R$  is a subset of pairs**

$R \subseteq A \times B, (a, b) \in R$  where  $a \in A$  and  $b \in B$ .

**ii)  $R$  is a infix binary boolean(relational) operator**

$R: A \times B \rightarrow \{\mathbf{T}, \mathbf{F}\}$

$a R b$  where  $a \in A$  and  $b \in B$ .

**iii)  $R$  is a set valued function from  $A$  to  $2^B$ .**

$R: A \rightarrow 2^B$ .

$R(a) = \{b_1, b_2, \dots, b_n\}$  where  $a \in A$  and  $\{b_1, b_2, \dots, b_n\} \in 2^B$ .

if  $1 \leq \forall i \leq n, (a, b_i) \in R$  or  $a R b_i$  for  $n \geq 0$ ,

if  $n=0, R(a) = \{b_1, b_2, \dots, b_n\} = \emptyset$ .

Note that  $\forall a \in A, \exists^1 \{b_1, b_2, \dots, b_n\} \in 2^B$  is unique.

### Three notations for the relation

i) **subset** of  $A \times B$ ,  $(a, b) \in R$ .

$$R \subseteq A \times B$$

ii) **infix binary boolean operation**,  $a R b$ ,

$$R: A \times B \rightarrow \{\mathbf{T}, \mathbf{F}\}, a R b.$$

*Example*  $=, <, \leq, \dots \subseteq N \times N$  or  $N \times N \rightarrow \{\mathbf{T}, \mathbf{F}\}$

$$3 = 3, 3 \neq 4, 3 \leq 3, \dots$$

iii) **set valued function**,  $R(a) = \{b_1, b_2, \dots, b_n\}$  or  $\emptyset$ .

$$R: A \rightarrow 2^B.$$

## Relation on A Set

**Def. 2** Let  $A$  be a set and  $R \subseteq A \times A$ .  $R$  is called a relation on  $A$ .

*Relation on  $A$  is a directed graph with vertices  $A$  and edges  $R$ .*

## Properties of Relations

**Def. 3** A relation  $R$  is **reflexive**, if  $\forall a \in A, a R a$ .

A relation  $R$  is **irreflexive**, if  $\forall a \in A, a \not R a$ .

*A relation may be neither reflexive nor irreflexive.*

**Def. 4** A relation  $R$  is **symmetric**, if  $a R b \Rightarrow b R a$ .

A relation  $R$  is **asymmetric**, if  $a R b \Rightarrow b \not R a$ .

A relation  $R$  is **antisymmetric**, if  $(a R b \wedge b R a) \Rightarrow (a = b)$ .

or if  $(a R b \wedge a \neq b) \Rightarrow b \not R a$ .

*If a relation is asymmetric then it is also antisymmetric. ( $\subseteq$ )*

**Def. 5** A relation  $R$  is **transitive**, if  $a R b \wedge b R c \Rightarrow a R c$ .

## Combining Relations

Let  $R_1, R_2 \subseteq A \times B$ . Consider  $R_1 \cup R_2, R_1 \cap R_2, R_1 \oplus R_2, R_1 - R_2, R_2 - R_1$ .

**Def. 6** Let  $R \subseteq A \times B, S \subseteq B \times C$ . Then **composition** of  $R$  and  $S$ , denoted as  $S \circ R = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$ .

**Def. 7** Let  $R \subseteq A \times A$ . Then for  $n \geq 1$ ,

$$R^1 = R \quad \text{basis}$$

$$R^{n+1} = R^n \circ R. \quad \text{induction}$$

**Def. 6.1** Let  $A$  be a set. We define an **identity relation**

$$id_A = \{(a, a) \in A \times A \mid a \in A\} = \Delta \text{ in this text sec. 9.4 in p. 977}$$

**Col. 0.5** Let  $R \subseteq A \times B$ . Then

$$R \circ id_A = id_A \circ R = R. \therefore id_A \text{ is a } \mathbf{identity} \text{ element for } \mathbf{composition}.$$

**Def. 7.1** Let  $R \subseteq A \times A$ . Then for  $n \in \mathbf{N}$ ,

$$R^0 = id_A \quad \text{basis} \quad (x^0 = 1)$$

$$R^{n+1} = R^n \circ R. \quad \text{induction}$$

**Thm. 1** Let  $R \subseteq A \times A$ .  $R$  is **transitive**, if and only if,  $R^n \subseteq R$  for  $\forall n \geq 1$ .

**Proof:**

1. (if)  $R^n \subseteq R$  for  $\forall n \geq 1 \Rightarrow R$  is **transitive**.

Since  $R^2 \subseteq R$ . If  $(a, b) \in R$  and  $(b, c) \in R$ ,  $(a, c) \in R^2$ .  $\therefore (a, c) \in R$   
 $\therefore R$  is **transitive**.

2. (only if)  $R$  is **transitive**  $\Rightarrow R^n \subseteq R$  for  $\forall n \geq 1$ .

mathematical induction on  $n \in \mathbf{N}^+$ .

**basis** Trivial for  $n = 1$  (since  $R^1 = R$ , by definition).

**induction** Assume  $R^n \subseteq R$  for some  $n \in \mathbf{N}^+$  and  $R$  is **transitive**.

Consider  $(a, b) \in R^{n+1}$ , then  $\exists x \in A$  . $\exists$ .  $(a, x) \in R$  and  $(x, b) \in R^n$ .

$R^n \subseteq R$ ,  $\therefore (x, b) \in R$ .      **induction hypothesis**

Since  $R$  is **transitive**;  $(a, x) \in R$  and  $(x, b) \in R \Rightarrow (a, b) \in R$ .

$\therefore R^{n+1} \subseteq R$ .

## 9.2 *n*-ary Relations and Their Applications

**Def. 1** Let  $A_1, A_2, \dots, A_n$  be sets.  $R \subseteq A_1 \times A_2 \times \dots \times A_n$  is a ***n*-ary relation** on  $A_1 \times A_2 \times \dots \times A_n$ . The sets  $A_1, A_2, \dots, A_n$  are called the **domains** of the relations, and  $n$  is called as the **degree** of  $R$ .

### Database and Relations

**relational data model:**  $R \subseteq A = A_1 \times A_2 \times \dots \times A_n$  is a **data base**

**record**  $(a_1, a_2, \dots, a_n) \in R,$

**field**  $1 \leq \forall f \leq n: a_f \in A_f$

**(primary) key field:**  $1 \leq \exists k \leq n: \forall K \in A_k . \exists . (a_1, a_2, \dots, K, \dots, a_n) \in R,$   
 $|(a_1, a_2, \dots, K, \dots, a_n)| \leq 1.$

The database  $R$  is said to be **functional** in the (**key**) field  $A_k$ .

*Example Table1 Students (St\_name, St\_id, Major, GPA) p. 564*



## Operations on $n$ -ary Relations

Let  $R \subseteq A = A_1 \times A_2 \times \dots \times A_n$ . Then

**Def. 2 Selection operator:** Let  $C: A \rightarrow \{\mathbf{T}, \mathbf{F}\}$ . Then

$$S_C(R) = \{a \in R \mid C(a), a \in A\}.$$

$C$ : condition       $S_C$ : selection operator on  $R$

**Def. 3 Projection Operator:**  $P_{i_1 i_2 \dots i_m}$  where  $i_1 < i_2 < \dots < i_m \leq n$  maps

$n$ -tuple  $(a_1, a_2, \dots, a_n) \in R$ , to the  $m$ -tuple  $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$  ( $i_m \leq n$ ).

$P_{\{i_k\}}: A \rightarrow A_{i_1} \times A_{i_2} \times \dots \times A_{i_m}$ , if  $\{i_k\} = (i_1, i_2, \dots, i_m)$  for  $1 \leq k \leq m$ .

$$P_{\{i_k\}}(a_1, a_2, \dots, a_n) = (a_{i_1}, a_{i_2}, \dots, a_{i_m}) \quad i_m \leq n.$$

**Def. 4 Join Operator:**

Let  $R_1 \subseteq A_1 \times \dots \times A_{m-p} \times C_1 \times \dots \times C_p$  and  $R_2 \subseteq C_1 \times \dots \times C_p \times B_1 \times \dots \times B_{n-p}$ .

Then  $J(R_1, R_2) \in A_1 \times \dots \times A_{m-p} \times C_1 \times \dots \times C_p \times B_1 \times \dots \times B_{n-p}$ .

If  $R_1$ :  $m$ -tuples,  $R_2$ :  $n$ -tuples, then  $J(R_1, R_2)$ :  $m+n-p$ -tuples. ( $p \leq m, n$ )

### 9.3 Representing Relations

#### Representing Relation as a Boolean Matrix

$R: A \times B \rightarrow \{0, 1\}$ . Let  $A = \{a_1, a_2, \dots, a_m\}$  and  $B = \{b_1, b_2, \dots, b_n\}$ .

A **boolean matrix**  $M_R = [m_{ij}]$  for the relation  $R$  where

$$1 \leq \forall i \leq m: 1 \leq \forall j \leq n:$$

$$m_{ij} = 1, \text{ if } (a_i, b_j) \in R,$$

$$m_{ij} = 0, \text{ if } (a_i, b_j) \notin R.$$

We may write  $M_R^{A \times B} [m_{ij}]$  instead of  $M_R$ , if needed.

#### Representing Relation as a Directed Graph(Digraph)

**Def. 1** A **directed graph** or **digraph**  $G = (V, E)$  consists of a set  $V$  of **vertices**, and a set  $E \subseteq V \times V$  of **edges(arcs)**.

$$R \subseteq A \times A. \quad \Leftrightarrow \quad G = (A, R)$$

**relation**  $R$  on  $A$  vs. **digraph** with vertices  $A$  and edges  $R$

## 9.4 Closure of Relations

**Closures** Let  $R \subseteq A \times A$  and  $\mathbf{P} = \{\text{reflexive, symmetric, transitive}\}$ . Then

We define a **p closure** of  $R$  as  $S \subseteq A \times A$  . $\exists$ . (1)  $R \subseteq S$  and

$\forall T \subseteq A \times A$  has the property  $\mathbf{p} \in \mathbf{P}$ ,  $R \subseteq T$ , and  $S \subseteq T$ .

(1)  $R \subseteq S$  and (2) **smallest one** among  $T$ 's( $\mathbf{p}$ )).

**Reflexive closure of  $R$**   $R \cup id_A$ .

$id_A = \Delta$  in the text( $\Delta_A$ ) **diagonal relation on  $A$ .**

**Symetric closure of  $R$**   $R \cup R^{-1}$ .

### Path in Digraph

**Def. 1** Let  $G = (V, E)$  be a digrph. For  $a, b \in V$ , a **path** from  $a, b$ ,

$Path(a, b) = (x_0, \underline{x_1}), (\underline{x_1}, x_2), \dots, (x_{n-1}, x_n)$

$1 \leq i \leq n: (x_{i-1}, x_i) \in E$ , and  $x_0 = a, x_n = b$ .

a **sequence of edges of length  $n$**

The path may be also denoted as a **sequence** of vertices

$$\text{Path}^V(a,b) = (x_0, x_1, x_2, \dots, x_n) \text{ of length } n.$$

We view the set of **empty** edges as a path of **length** 0 from  $a \in V$  to  $a$ .

A path of length  $n \geq 1$  that

**begins and ends at the same vertex is called cycle.**

**Thm. 1** Let  $R \subseteq A \times A$ . There is a **path** of length  $n \geq 1$  from  $a$  to  $b$ ,

**if and only if,  $(a, b) \in R^n$ .**

**proof** easy (mathematical induction)

**Def. 1** A **connectivity** relation  $R^+ = \{(a, b) \mid (a, b) \in R^n, \forall n \geq 1\}$

$$R^+ = \cup_{i \in \mathbf{N}_1} R^i = R^1 \cup R^2 \cup \dots$$

**transitive closure of  $R$ .**

$$R^* = \cup_{i \in \mathbf{N}_0} R^i = R^0 \cup R^1 \cup R^2 \cup \dots$$

**reflexive and transitive closure of  $R$ .**

### Computing transitive closure $R^*$

Let  $A$  and  $B$  be sets,  $R \subseteq A \times A$ ,  $f, g: A \rightarrow 2^C$ . (set valued functions), and

$$f(a) = \{c \in C \mid c \in g(a)\} \cup \{c \in C \mid a R b, c \in f(b)\}.$$

$$f(a) = \{c \in g(a)\} \cup \{c \in f(b) \mid a R b\}.$$

$$f(a) = g(a) \cup \cup_{a R b} f(b). \text{ (recursive definition of } f) \text{ Then}$$

$$f(a) = \{c \in C \mid c \in g(b), a R^* b\}.$$

$$f(a) = g^*(a). \text{ (iterative definition of } f)$$

Warshall's algorithm  $O(n^3)$

Depth first search  $O(n^2)$

**Algorithm** *Depth first search*

*S*: stack of Vertex; *n*(Vertex) array of Depth;

**procedure** *Traverse*(*x*: Vertex; *d*: Depth);

**push** *x* onto *S*;  $n(x) := d$ ;

$f(x) := g(x)$ ;

**for**  $y \in \text{Vertex}$  **where**  $x R y$  **do**

**if**  $n(y) = 0$  **then** *Traverse*(*y*, *d*+1) **fi**;

$n(x) := \min(n(x), n(y))$ ;

$f(x) := f(x) \cup f(y)$

**od**;

**if**  $n(x) = d$  **then repeat**

$y = \text{pop of } S$ ;  $n(y) := \text{infinite}$ ;

$f(y) := f(x)$

**until**  $y = x$

**fi**

**end procedure** *Traverse*

**for**  $x \in \text{Vertex}$  **do**  $n(x) := 0$ ;

$f(x) := \{\}$  **od**;

**for**  $x \in \text{Vertex}$  **where**  $n(x) = 0$  **do** *Traverse*(*x*, 1) **od**

## 9.5 Equivalence Relations

**Def. 1** Let  $R \subseteq A \times A$ .  $R$  is called **equivalence relation**, if it is reflexive, symmetric, and transitive.

**Def. 2**  $a, b \in A$  are said to be related **equivalent**, written  $a \sim b$ , if  $R$  is an equivalent relation on  $A$  and  $a R b$ .

**Def. 3** Let  $R \subseteq A \times A$  be an **equivalence relation**.

$[a]_R = \{b \mid a R b\}$  is called the **equivalence class** of  $a$  w.r.t.  $R$ .

If  $b \in [a]_R$ ,  $b$  is called the **representative** of the equivalent class.

Note that  $a \in [a]_R$ , since  $R$  is reflexive.

**Thm. 1** Let  $R \subseteq A \times A$  be an **equivalence relation**. Three statements are logically equivalent

$$i) a R b \quad ii) [a]_R = [b]_R \quad iii) [a]_R \cap [b]_R \neq \emptyset.$$

**proof**1) i)  $\rightarrow$  ii)

$$\forall c \in [a]_R, a R c, a R b, b R a. \therefore b R c, c \in [b]_R. \therefore [a]_R \subseteq [b]_R.$$

$$\forall c \in [b]_R, b R c, a R b. \therefore a R c, c \in [a]_R. \therefore [b]_R \subseteq [a]_R.$$

2) ii)  $\rightarrow$  iii)

$$\text{Assume } [a]_R = [b]_R, a R b. \therefore a, b \in [a]_R \cap [b]_R \neq \emptyset.$$

3) iii)  $\rightarrow$  i)

$$\text{Suppose } [a]_R \cap [b]_R \neq \emptyset, [a]_R \neq \emptyset \text{ and } [b]_R \neq \emptyset.$$

$$\exists c \in [a]_R \wedge c \in [b]_R. a R c, b R c. c R b, \therefore a R b.$$

**Lemma 1.5** Let  $R \subseteq A \times A$  be an **equivalence relation**.

$$\cup_{a \in A} [a]_R = A. \quad a \in [a]_R.$$

$\therefore \{[a]_R \subseteq A \mid a \in A\}$  is a **partition** of  $A$ .



**Definition 1.5** Let  $S$  be a set. The **partition** of  $S$ ,  $\{A_i \mid i \in I\}$   $I$ : index set, is

- i)  $A_i \neq \emptyset, i \in I.$  **nonempty**
- ii)  $A_i \cap A_j = \emptyset,$  when  $i \neq j.$  **disjoint**
- iii)  $\cup_{i \in I} A_i = A.$  **exhaustive**

**Thm. 2** Let  $R$  be a equivalent relation on  $A$ .

Then the **equivalent classes** of  $R$  form a **partition** of  $A$ .

Conversely, given a **partition**  $\{A_i \mid i \in I\}$  of the set  $A$ ,

there is an **equivalent relation**  $R$  that has the set  $A_i, i \in I,$   
as its **equivalent class**.

**Relation**  $R \subseteq A \times A$   $O(n^2)$  where  $|A| = n.$

**Equivalent relation**  $R \subseteq A \times A$   $O(n)$

## 9.6 Partial Ordering

**Def. 1** Let  $R \subseteq A \times A$ .  $R$  is called **(ir)reflexive partial order**, if it is **(ir)reflexive**, **antisymmetric**, and **transitive**.

$(A, R)$  is called **partially ordered set or poset**.

**Ex. 1 2 3**  $(\mathbb{Z}, \leq)$ ,  $(\mathbb{Z}^+, |)$ ,  $(2S, \subseteq)$  are **posets**.

**Def. 2** Let  $(A, \leq)$  be poset and  $a, b \in A$ . Then

The elements  $a$  and  $b$  are **comparable** if either  $a \leq b$  or  $b \leq a$ .

The elements  $a$  and  $b$  are **incomparable** if **neither**  $a \leq b$  **nor**  $b \leq a$ .

**Def. 3** Let  $(S, \leq)$  be poset. If  $\forall a, b \in S$ ,  $a$  and  $b$  are **comparable**,  $S$  is called **totally ordered set**, **linearly ordered set**, or **chain**.  $\leq$  is called **total order** or **linear order**.

**Ex. 6, 7**  $(\mathbb{Z}, \leq)$  is a **total order**, **but**  $(\mathbb{Z}^+, |)$  is **not** a **total order**

**Def. 4** A poset  $(S, \leq)$  is a **well-ordered set**, if  $\leq$  is a **total order**, and every nonempty subset of  $S$  has a **least element**.

**Ex. 8**  $(\mathbf{Z}^+ \times \mathbf{Z}^+, \leq_L)$  is **well-ordered**, **but**  $(\mathbf{Z}, \leq)$  is **not** a well-order

### **Thm .1 Principle of Well-Ordered Induction**

Let  $(S, \leq)$  be a well-ordered set and  $\forall x \in S, P(x)$ , if

$$\forall y \in S .\exists. x < y: P(x) \Rightarrow P(y).$$

**proof** Suppose  $\exists y \in S, \neg P(y)$ .

$$A = \{x \in S \mid \neg P(x)\} \neq \emptyset.$$

Let  $a \in A$  be the least element.

$$\exists a \in A, \forall x \in S .\exists. x < a: P(x) \Rightarrow P(a).$$

**Contradiction**  $\exists y \in S, \neg P(y)$ .

### Lexicographic order( 사전순서 )

Let  $(A_1, \leq_1)$  and  $(A_2, \leq_2)$  be **well-ordered sets**.

We define  $(A_1 \times A_2, \leq_{1 \times 2} = \leq_L)$  lexicographic order

$$(a_1, a_2) \leq_{1 \times 2} (b_1, b_2), \text{ if } (a_1 <_1 b_1) \vee ((a_1 =_1 b_1) \wedge (a_2 \leq_2 b_2)).$$

**Ext.** Let  $(A_1, \leq_1), (A_2, \leq_2), \dots, (A_n, \leq_n)$  be well-ordered sets. Then

We define  $(A_1 \times A_2 \times \dots \times A_n, \leq_{1 \times 2 \dots \times n} = \leq_L)$  lexicographic order

$$(a_1, a_2, \dots, a_n) \leq_L (b_1, b_2, \dots, b_n), \text{ if } (1 \leq \exists i < n: a_i =_i b_i) \wedge (a_i <_i b_i).$$

**Def.** Let  $(V, \leq)$  is a well-ordered set. Then We define  $(V^n, \leq_L)$

$$(a_1, a_2, \dots, a_n) \leq_n (b_1, b_2, \dots, b_n), \text{ if } (1 \leq \exists i < n: a_i = b_i) \wedge (a_i < b_i).$$

**Ext.** Let  $(V, \leq)$  is a well-ordered set. Then We define  $(V^*, \leq_L)$

lexicographic order as

$$x, y \in V^*, x \leq_L y, \text{ if } (|x| < |y|) \vee [ |x| = |y| \wedge (x \leq_n y)].$$

$(V^*, \leq_L)$  is a **well-ordered set**.

## Hasse Diagram

Let  $(A, \leq)$  be a poset. We say  $a$  **covers**  $b$ , if  $a < b \wedge \neg \exists c \in A, a \leq c \wedge c \leq b$ .  
 $(A, \text{covers})$  is a Hasse Diagram.

## Maximal and Minimal Elements

Let  $(A, \leq)$  be a poset.

We say  $a$  is a **maximal**, if  $\neg \exists b \in A, a < b$ .

We say  $a$  is a **minimal**, if  $\neg \exists b \in A, b < a$ .

We say  $a$  is a **greatest element**, if  $\forall b \in A, b \leq a$ .

We say  $a$  is a **least elements**, if  $\forall b \in A, a \leq b$ .

Let  $S \subseteq A$ .  $u \in A$  is called **upper bound** of  $S$ , if  $\forall a \in S, a \leq u$ .

$l \in A$  is called **lower bound** of  $S$ , if  $\forall a \in S, l \leq a$ .

$x \in A$  is called the **least upper bound** of  $S$ , if  $\forall a \in S, a \leq x$ ,

$x \leq z$ , if  $\forall z \in \text{upper bound of } S$ .

$y \in A$  is called the **greatest lower bound** of  $S$ , if  $\forall a \in S, y \leq a$ ,

$z \leq y$ , if  $\forall z \in \text{lower bound of } S$ .

**Lattice**

A poset  $(A, \leq)$  is called **lattice**. If every pair of elements has both a least upper bound, and a greatest lower bound.

$(\mathbf{Z}, \leq)$  is a lattice.

$\max, \min: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$

$(\mathbf{Z}^+, |)$  is a lattice.

$\text{lcm}, \text{gcd}: \mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$

$(2^S, \subseteq)$  is a lattice.

$\cup, \cap: 2^S \times 2^S \rightarrow 2^S$

**Topological Sorting**

**Lemma 1** Every finite nonempty poset  $(A, \leq_p)$  has  
at least one minimal elements.

**Algorithm 1 Topological sorting**

Construct a total ordering  $<_t$  from a partial ordering  $\leq_p$ .

$a <_t b$ , if and only if,  $a \leq_p b$  or  $a$  and  $b$  are **incomparable**.