9 Relations

9.1 Relations and Their Properties

Def. *1 Let A and B be two set. Then a* ***binary relation*** *R from A to B is subset of* $A \times B$.

$$R \subseteq A \times B.$$

A: ***domain*** *of the relation R. B:* ***range(codomain)*** *of the relation R.*

Let $a \in A$, $b \in B$, *Then* $(a, b) \in R$ *or* $(a, b) \notin R$.

If $(a, b) \in R$, *we also write a R b and we say a is* ***related to*** *b by R.*

If $(a, b) \notin R$, *we also write* $a \not R b$ *and a is* ***not related to*** *b by R.*

Two ***aspects*** *of Relation*

$R \subseteq A \times B$ *relation R is a* ***set*** *of pairs*

$(a, b) \in R$

$R: A \times B \to \{\mathbf{T}, \mathbf{F}\}$ *relation R is a infix* ***boolean*** *binary* ***operation***

$a\ R\ b$

Ex. $(3, 3) \in =$ *and* $(3, 4) \notin =$; *or* $3 = 3$ *and* $3 \neq 4$.

***Function is a Relation* but relation is not a function!**

$R \subseteq A \times B$ vs $f: A \rightarrow B$ $\forall a \in A: \exists_1 f(a)=b \in B$.

If $f(a) = b$, *then we can write* $(a, b) \in f$ *or* $a\ f\ b$.

. $\therefore$ *Function is a (**special** kind of) relation.*

*Relation is **not** a function*

i) *If* $\exists a \in A$.∋. $a\ R\ b_1$ *and* $a\ R\ b_2$,$(b_1 \neq b_2)$, *R is **not** a function.*

*function must have an unique **related** image.*

ii) *If* $\exists a \in A$ *and* $\nexists b \in B$.∋. $a\ R\ b$, *R is **not** a function.*

all the elements in the domain must have its related image

Let $R \subseteq A \times B$. *Then*

We write $R(a) = \{b_1, b_2, \ldots, b_n\} \in R$, *if* $1 \leq {}^{\forall} i \leq n: (a, b_i) \in R$.

A ***function*** *is a* ***special kind*****(*****subclass*****)** *of a* ***relation***

Three *faces of the relation R form A to B.*

*i) R is a sub**set of pairs***

$R \subseteq A \times B$, $(a, b) \in R$ where $a \in A$ and $b \in B$.

ii) R is a ***infix binary boolean(relational) operator***

$R: A \times B \rightarrow \{\mathbf{T}, \mathbf{F}\}$

$a\ R\ b$ where $a \in A$ and $b \in B$.

iii) R is a ***set valued function*** *from A to* 2^B.

$R: A \rightarrow 2^B$.

$R(a) = \{b_1, b_2, \ldots, b_n\}$ where $a \in A$ and $\{b_1, b_2, \ldots, b_n\} \in 2^B$.

if $1 \le {}^{\forall} i \le n$, $(a, b_i) \in R$ or $a\ R\ b_i$. for $n \ge 0$,

if $n=0$, $R(a) = \{b_1, b_2, \ldots, b_n\} = \emptyset$.

Note that ${}^{\forall} a \in A$, ${}^{\exists 1}\{b_1, b_2, \ldots, b_n\} \in 2^B$ is unique.

Three notations for the relation

i) ***subset*** *of* $A \times B$, $(a, b) \in R$.

$R \subseteq A \times B$

ii) infix binary boolean ***operation***, $a\ R\ b$,

$R: A \times B \to \{\mathbf{T}, \mathbf{F}\}$, $a\ R\ b$.

Example $=, <, \leq, \ldots \subseteq N \times N$ *or* $N \times N \to \{\mathbf{T}, \mathbf{F}\}$

$3 = 3, 3 \neq 4, 3 \leq 3, \ldots$

iii) ***set valued function***, $R(a) = \{b_1, b_2, \ldots, b_n\}$ *or* $\varnothing$.

$R: A \to 2^B$.

Relation on A Set

Def. *2 Let A be a set and* $R \subseteq A \times A$. *R is called a relation* **on** *A.*

Relation on A is a ***directed graph*** *with* ***vertices*** *A and* ***edges*** *R.*

Properties of Relations

Def. *3 A relation R is* ***reflexive****, if* $\forall a \in A$, $a\,R\,a$.

A relation R is ***irreflexive****, if* $\forall a \in A$, $a\,\not R\,a$.

A relation may be ***neither*** *reflexive* ***nor*** *irreflexive.*

Def. *4 A relation R is* ***symmetric****, if* $a\,R\,b \Rightarrow b\,R\,a$.

A relation R is ***asymmetric****, if* $a\,R\,b \Rightarrow b\,\not R\,a$.

A relation R is ***antisymmetric****, if* $(a\,R\,b \wedge b\,R\,a) \Rightarrow (a = b)$.

or if $(a\,R\,b \wedge a \neq b) \Rightarrow b\,\not R\,a$.

If a relation is a ***asymmetric*** *then it is also* ***antisymmetric****.*$(\subseteq)$

Def. *5 A relation R is* ***transitive****, if* $a\,R\,b \wedge b\,R\,c \Rightarrow a\,R\,c$.

Combining Relations

Let $R_1, R_2 \subseteq A \times B$. *Consider* $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, $R_2 - R_1$.

Def. 6 *Let* $R \subseteq A \times B$, $S \subseteq B \times C$. *Then* ***composition*** *of R and S, denoted as* $S \circ R = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$.

Def. 7 *Let* $R \subseteq A \times A$. *Then for* $n \in \mathbf{N}^+$,

$R^1 = R$ ***basis***

$R^{n+1} = R^n \circ R$. ***induction***

Def. 6.1 *Let A be a set. We define* ***identity*** *relation*

$id_A = \{(a, a) \in A \times A \mid a \in A\}$

Colorally *0.5 Let* $R \subseteq A \times B$. *Then*

$R \circ id_A = id_A \circ R = R$. id_A *is a* ***identity*** *element for* ***composition***.

Def. 7.1 *Let* $R \subseteq A \times A$. *Then for* $n \in \mathbf{N}$,

$R^0 = id_A$ ***basis*** $(x^0 = 1)$

$R^{n+1} = R^n \circ R$. ***induction***

Thm. 1 *Let $R \subseteq A \times A$. R is* ***transitive****, if and only if, $R^n \subseteq R$ for $\forall n \in \mathbf{N}^+$.*

Proof*:*

1. (if) $R^n \subseteq R$ for $\forall n \in \mathbf{N}^+ \Rightarrow R$ is transitive.

Since $R^2 \subseteq R$. If $(a, b) \in R$ and $(b, c) \in R$, $(a, c) \in R^2$. $\therefore$ $(a, c) \in R$

$\therefore$ R is transitive.

2.(only if) R is transitive $\Rightarrow R^n \subseteq R$ for $\forall n \in \mathbf{N}^+$.

mathematical induction on $n \in \mathbf{N}^+$.

basis *Trivial for n = 1.*

induction *Assume $R^n \subseteq R$ for some $n \in \mathbf{N}^+$ and R is* ***transitive****.*

Consider $(a, b) \in R^{n+1}$, then $\exists x \in A$.э. $(a, x) \in R$ and $(x, b) \in R^n$.

$R^n \subseteq R$, $(x, b) \in R$. ***induction hypothesis***

Since $R^{n+1} = R^n \circ R$ and R is transitive, $(a, b) \in R$.

$\therefore$ $R^{n+1} \subseteq R$.

9.2 n-ary Relations and Their Applications

Def. *1 Let* $A_1, A_2, \ldots, A_n$ *be sets.* $R \subseteq A_1 \times A_2 \times \ldots \times A_n$ *is a* ***n-ary relation*** *on* $A_1 \times A_2 \times \ldots \times A_n$. *The sets* $A_1, A_2, \ldots, A_n$ *are called the* ***domains*** *of the relations, and n is called as the* ***degree*** *of R.*

Database and Relations

relational *data model:* $R \subseteq A = A_1 \times A_2 \times \ldots \times A_n$ *is a* ***data base***

record $(a_1, a_2, \ldots, a_n) \in R$,

field $1 \le {}^{\forall}f \le n: a_f \in A_f$.

(***primary***) ***key*** *field:* $1 \le {}^{\exists}k \le n: {}^{\forall}K \in A_k$.э. $(a_1, a_2, \ldots, K, \ldots, a_n) \in R$,

$$|(a_1, a_2, \ldots, K, \ldots, a_n)| \le 1.$$

The ***dadabase*** *R is said to be* ***functional*** *in the* (***key***) *field* A_k.

Examle Table1 Students (St_name, St_id, Major, GPA) p. 564

Operations on n-ary Relations

Let $R \subseteq A = A_1 \times A_2 \times \ldots \times A_n$. *Then*

Def. 2 **Selection** *operator: Let* $C: A \rightarrow \{\mathbf{T}, \mathbf{F}\}$. *Then*

$$S_C(R) = \{a \in R \mid C(a), a \in A\}.$$

C: *condition* $\quad S_C$: *selection operator on R*

Def. 3 **Projection** *Operator*: $P_{i_1 i_2 \ldots i_m}$ *where* $i_1 < i_2 < \ldots < i_m \leq n$ *maps n-tuple* $(a_1, a_2, \ldots, a_n) \in R$, *to the m-tuple* $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})(i_m \leq n)$.

$P_{\{i_k\}}: A \rightarrow A_{i_1} \times A_{i_2} \times \ldots \times A_{i_m}$, *if* $\{i_k\} = (i_1, i_2, \ldots, i_m)$ *for* $1 \leq k \leq m$.

$$P_{\{i_k\}}(a_1, a_2, \ldots, a_n) = (a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \qquad i_m \leq n.$$

Def. 4 **Join** *Operator:*

Let $R_1 \subseteq A_1 \times \ldots \times A_{m-p} \times C_1 \times \ldots \times C_p$ *and* $R_2 \subseteq C_1 \times \ldots \times C_p \times B_1 \times \ldots \times B_{n-p}$.

Then $J(R_1, R_2) \in A_1 \times \ldots \times A_{m-p} \times C_1 \times \ldots \times C_p \times B_1 \times \ldots \times B_{n-p}$.

If R_1: *m-tuples*, R_2: *n-tuples, then* $J(R_1, R_2)$: $m+n-p$*-tuples.* $(p \leq m, n)$

9.3 Representing Relations

Representing Relation as a Boolean Matrix

$R: A \times B \rightarrow \{0, 1\}$. Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

A **boolean matrix** $M_R = [m_{ij}]$ for the relation R where

$1 \leq {}^{\forall}i \leq m: 1 \leq {}^{\forall}j \leq n:$

$m_{ij} = 1$, if $(a_i, b_j) \in R$,

$m_{ij} = 0$, if $(a_i, b_j) \notin R$.

We may write $M_R^{A \times B}[m_{ij}]$ **instead** of M_R, if needed.

Representing Relation as a Directed Graph(Digraph)

$R \subseteq A \times A$. $G = (A, R)$

relation on A vs. **digraph** with vertices A and edges R

Def. 1 A **directed graph** or **digraph** $G = (V, E)$ consistes of a set V of **vertices**, and a set $E \subseteq V \times V$ of **edges(arcs)**.

9.4 Closure of Relations

Let $R \subseteq A \times A$. R may or may not have some property **P** *={reflextive, symmetric, transitive}. If $\forall T \subseteq A \times A$ with property* **p** $\in$ **P** *and $R \subseteq T$, $S \subseteq T$, then S is called the* **p** ***closue*** *of R.*

Reflexive *closure of R*

$R \cup id_A$. $id_A = \Delta$ *in the text*(Δ_A) ***diagonal*** *relation on A.*

Symetric *closure of R*

$R \cup R^{-1}$.

***Def**. 1 A **path** from a to b in the directed grapg G = (V, E).*

$(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n) \in E$, and $x_0 = a$, $x_n = b$.

The path is denoted by $(x_0, x_1, x_2, \ldots, x_n)$ *and has **length** n.*

A path of length $n \geq 1$ and

*begins and ends at the same vertex is called **cycle**.*

Thm. *1 Let* $R \subseteq A \times A$. *There is a path of length* $n \geq 1$ *from a to b,*

if and only if, $(a, b) \in R^n$.

proof *Easy for induction*

Def. *1 A* ***connetivity*** *relation* $R^+ = \{(a, b) | (a, b) \in R^n, \forall n \geq 1\}$

$$R^+ = \cup_{i \in \mathbf{N}^+} R^i = R^1 \cup R^2 \cup \ldots$$

transitive *closure of R.*

$$R^* = \cup_{i \in \mathbf{N}} R^i = R^0 \cup R^1 \cup R^2 \cup \ldots$$

reflexive and transitive *closure of R.*

Computing transitive closure R^*.

Let A and B be sets, $R \subseteq A \times A$, $f, g: A \rightarrow 2^C$.*(set valued functions), and*

$f(a) = \{c \in C \mid c \in g(a)\} \cup \{c \in C \mid a\ R\ b,\ c \in f(b)\}$.

$f(a) = \{c \in g(a)\} \cup \{c \in f(b) \mid a\ R\ b\}$.

$f(a) = g(a) \cup \cup_{a\ R\ b} f(b)$. (**recursive** *definition of f*) *Then*

$f(a) = \{c \in C \mid c \in g(b),\ a\ R^*\ b\}$.

$f(a) = g^*(a)$. (**iterative** *definition of f*)

Warshall's algorithm	$O(n^3)$
Depth first search	$O(n^2)$

Algorithm *Depth first search*

```
S: stack of Vertex; n(Vertex) array of Depth;
procedure Traverse(x: Vertex; d: Depth);
    push x onto S; n(x) := d;                    f(x) := g(x);
    for y ∈ Vertex where x R y do
        if n(y) = 0 then Traverse(y, d+1) fi;
        n(x) := min(n(x), n(y));                 f(x) := f(x) ∪ f(y)
        od;
    if n(x) = d then repeat
        y = pop of S; n(y) := infinite;          f(y) := f(x)
        until  y = x
    fi
end procedure Traverse
for x  ∈ Vertex do n(x) := 0;                    f(x) := {} od;
for x  ∈ Vertex where n(x) = 0 do Traverse(x, 1) od
```

9.5 Equivalence Relations

Def. *1 Let $R \subseteq A \times A$. R is called* ***equivalence*** *relation,*
if it is reflexive, symmetric, and transitive.

Def. *2 $a, b \in A$ are said to be related* ***equivalent***, *written $a \sim b$, if R is an equivalent relation on A and a R b.*

Def. *3 Let $R \subseteq A \times A$ be an* ***equivalence*** *relation.*
$[a]_R = \{b \mid a\ R\ b\}$ is called the ***equivalence class*** *of a w.r.t. R.*
If $b \in [a]_R$, b is called the ***representative*** *of the equivalent class.*
Note that $a \in [a]_R$, since R is ***reflexive***.

Thm. *1 Let $R \subseteq A \times A$ be an* ***quivalence*** *relation. Three statments are logically equivalent*
i) a R b *ii) $[a]_R = [b]_R$* *iii) $[a]_R \cap [b]_R \neq \emptyset$.*

proof

1) i) $\rightarrow$ ii)

$\forall c \in [a]_R$, $a\ R\ c$, $a\ R\ b$, $b\ R\ a$. $\therefore$ $b\ R\ c$, $c \in [b]_R$. $\therefore$ $[a]_R \subseteq [b]_R$.

$\forall c \in [b]_R$, $b\ R\ c$, $a\ R\ b$. $\therefore$ $a\ R\ c$, $c \in [a]_R$. $\therefore$ $[b]_R \subseteq [a]_R$.

2) ii) $\rightarrow$ iii)

Assume $[a]_R = [b]_R$, $a\ R\ b$. $\therefore$ $a, b \in [a]_R \cap [b]_R \neq \varnothing$.

3) iii) $\rightarrow$ i)

Suppose $[a]_R \cap [b]_R \neq \varnothing$, $[a]_R \neq \varnothing$ *and* $[b]_R \neq \varnothing$.

$\exists c \in [a]_R \wedge c \in [b]_R$. $a\ R\ c$, $b\ R\ c$. $c\ R\ b$, $\therefore$ $a\ R\ b$.

Lemma *1.5 Let* $R \subseteq A \times A$ *be an* ***equivalence*** *relation.*

$\cup_{a \in A} [a]_R = A$. $a \in [a]_R$.

$\therefore$ $\{[a]_R \subseteq A \mid a \in A\}$ *is a* ***partition*** *of* A.

Definition *1.5 Let S be a set. The* ***partition*** *of S,* $\{A_i \mid i \in I\}$ *I: index set, is*

- *i)* $A_i \neq \emptyset$, $i \in I$. ***nonempty***
- *ii)* $A_i \cap A_j = \emptyset$, *when* $i \neq j$. ***disjoint***
- *iii)* $\cup_{i \in I} A_i = A$. ***exhaustive***

Thm*. 2 Let R be a equivalent relation on A.*
Then the ***equivalent classes*** *of R form a* ***partion*** *of A.*
Conversely, given a ***partition*** $\{A_i \mid i \in I\}$ *of the set A,*
there is an ***equivalent relation*** *R that has the set* A_i, $i \in I$,
as its ***equivalent class****.*

Realtion $R \subseteq A \times A$ $O(n^2)$ *where* $|A| = n$.
Equivalent *relation* $R \subseteq A \times A$ $O(n)$

9.6 Partial Ordering

Def. 1 Let $R \subseteq A \times A$. R is called ***(ir)reflexive* partial order**,
if it is ***(ir)reflexive***, ***antisymmetric***, and ***transitive***.
(A, R) is called ***partially odreded*** set or ***poset***.

Ex. 1 2 3 $(\mathbf{Z}, \leq)$, $(\mathbf{Z}^+, |)$, $(2^S, \subseteq)$ are posets.

Def. 2 Let $(A, \leq)$ be poset and $a, b \in A$. Then
The elements a and b are ***comparable*** if either $a \leq b$ or $b \leq a$.
The elements a and b are ***incomparable*** if ***neither*** $a \leq b$ ***nor*** $b \leq a$.

Def. 3 Let $(S, \leq)$ be poset. If $\forall a, b \in S$, a and b are ***comparable***,
S is called ***totally ordered set***, ***linearly ordered set***, or ***chain***.
$\leq$ is called ***total order*** or ***linear order***.

***Def**. 4 $(S, \leq)$ is a **well-ordered** set, if it is a **poset**, $\leq$ is a **total order**,*
*and every nonempty subset of S has a **least** emelent.*

Thm** .1 **Principle of Well-Ordered Induction

Let $(S, \leq)$ be a well-ordered set. $\forall x \in S, P(x)$, if

$\forall y \in S, \forall x \in S$.϶. $x < y: P(x) \Rightarrow P(y)$.

***proof** Suppose $\exists y \in S, \neg P(y)$.*

$A = \{x \in S \mid \neg P(x)\} \neq \emptyset$.

Let $a \in A$ be the least element.

$\exists a \in A, \forall x \in S$.϶. $x < a: P(x) \Rightarrow P(a)$.

Contradiction $\exists y \in S, \neg P(y)$.

Lexicographic order

*Let $(A_1, \leq_1)$ and $(A_2, \leq_2)$ be **well-ordered** sets.*

We define $(A_1 \times A_2, \leq)$ lexicographic order

$(a_1, a_2) \leq (b_1, b_2)$, if $(a_1 <_1 b_1) \vee ((a_1 =_1 b_1) \wedge (a_2 \leq_2 b_2))$

Hasse Diagram

Let $(A, \leq)$ be a poset. We say a **covers** *b, if $a < b \wedge \neg\exists c \in A, a \leq c \wedge c \leq b$.*
(A, covers) is a Hasse Diagram.

Maximal and Minimal Elements

Let $(A, \leq)$ be a poset.
We say a is a **maximal**, *if $\neg\exists b \in A, a < b$.*
We say a is a **minimal**, *if $\neg\exists b \in A, b < a$.*
We say a is a **greatest element**, *if $\forall b \in A, b \leq a$.*
We say a is a **least elements**, *if $\forall b \in A, a \leq b$.*
Let $S \subseteq A$. $u \in A$ is called **upper bound** *of S, if $\forall a \in S, a \leq u$.*
$l \in A$ is called **lower bound** *of S, if $\forall a \in S, l \leq a$.*
$x \in A$ is called the **least upper bound** *of S, if $\forall a \in S, a \leq x$,*
$x \leq z$, if $\forall z \in$ upper bound of S.
$y \in A$ is called the **greatest lower bound** *of S, if $\forall a \in S, y \leq a$,*
$z \leq y$, if $\forall z \in$ lower bound of S.

Lattice

A poset (A, $\leq$) is called ***lattice****. If every pair of elements has both a least upper bound, and a greatest lower bound.*

($\mathbf{Z}$, $\leq$) *is a lattice.* *max, min:* $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$

($\mathbf{Z}^+$, $|$) *is a lattice.* *lcm, gcd:* $\mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$

(2^S, $\subseteq$) *is a lattice.* $\cup, \cap: 2^S \times 2^S \rightarrow 2^S$

Topological Sorting

Lemma *1 Every* ***finite nonempty*** *poset (A, $\leq_p$) has* ***at least one minimal*** *elements.*

Algorithm *1* ***Topological sorting***

Construct a total ordering $<_t$ from a partial ordering $\leq_p$.

$a <_t b$, if and only if, $a \leq_p b$ or a and b are ***incomparable****.*