

8 Advanced Counting Techniques

8.1 Application of Recurrence Relations

Def. 1 A recurrence relation for a sequence $\{a_n\}$ is

an equation expresses a_n in terms of $a_0, a_1, \dots, a_{n-1} \forall n \geq n_0$.

A sequence is said to be a **solution** of a recurrence, if it consistent with the definition of the recurrence.

Ex. 1 $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$ with $a_0 = 3$ and $a_1 = 5$.

Ex. 3 Compound Interests

$P_n = P_{n-1} + 0.11P_{n-1} = 1.11P_{n-1}$ for $n \geq 1$ with $P_0 = 1$.

$P_n = 1.11P_{n-1} = 1.11(1.11P_{n-2}) = 1.11^3P_{n-3} = \dots = 1.11^nP_0$.

Ex. 4 Rabbits and Fibonacci number

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with $f_0 = 0$ and $f_1 = 1$.

Ex. 5 Tower of Hanoi

1. move $(n-1)$ disks 2. move bottom 1 disk 3. move $(n-1)$ disks.

$$H_n = 2H_{n-1} + 1 \quad n \geq 2 \text{ with } H_1 = 1.$$

$$H_n = 2H_{n-1} + 1$$

$$= 2(2H_{n-2} + 1) = 2^2H_{n-2} + 2 + 1$$

$$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3H_{n-3} + 2^2 + 2 + 1$$

...

$$= 2^{n-k-1}H_{k+1} + 2^{n-k-2} + 2^{n-k-3} + \dots + 2^2 + 2 + 1$$

$$= 2^{n-k-1}(2H_k + 1) + 2^{n-k-2} + 2^{n-k-3} + \dots + 2^2 + 2 + 1$$

$$= 2^{n-k}H_k + 2^{n-k-1} + 2^{n-k-2} + 2^{n-k-3} + \dots + 2^2 + 2 + 1$$

...

$$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \dots + 2^2 + 2 + 1 \text{ (with } k = 1)$$

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2^2 + 2 + 1 = 2^n - 1.$$

8.2 Solving Linear Recurrence Relations

Def. 1 A linear homogeneous recurrence relation of degree k with constant coefficient is the recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \quad \forall n \geq k \text{ with } a_0, a_1, \dots, a_{k-1}.$$

where $c_1, c_2, \dots, c_k \in \mathbf{R}, c_k \neq 0$.

The solution is uniquely determined if k initial conditions a_0, a_1, \dots, a_{k-1} are provided.

Characteristic equation

Assume $a_n = r^n$.

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}, \text{ i.e.,}$$

$$r^k - c_1 r^{k-1} - \dots - c_{k-1} r - c_k = 0$$

k characteristic roots

Consider a recurrence relation of degree 2

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

Characteristic equation

$$r^2 - c_1 r - c_2 = 0$$

Thm. 1 If C.E. of *degree 2* has two **distinct** roots $r_1 \neq r_2$, **solution** is

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n, \text{ for } \forall \alpha_1, \alpha_2 \in \mathbf{R}.$$

proof Assume $a_n =_R \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution.

We know $r_1^2 =_C c_1 r_1 + c_2$ and $r_2^2 =_C c_1 r_2 + c_2$.

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= _R c_1 (\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2 (\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2} (c_1 r_1 + c_2) + \alpha_2 r_2^{n-2} (c_1 r_2 + c_2) \\ &= _C \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 = \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= _R a_n. \end{aligned}$$

Ex. 3.5 small Tower of Hanoi $h_n = 2h_{n-1}$ with $h_1 = 1$ (degree 1).

characteristic equation $r - 2 = 0, r = 2.$

$$h_n = \alpha 2^n. \quad h_1 = 2\alpha = 1 \quad \therefore \alpha = 2^{-1}.$$

$$\therefore h_n = 2^{n-1}. \quad \text{Note that } H_n = 2^n - 1.$$

Ex. 4 Fibonacci number $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0, f_1 = 1.$

characteristic equation $r^2 - r - 1 = 0.$

$$r_1 = (1+5^{1/2})/2, r_2 = (1-5^{1/2})/2.$$

$$f_n = \alpha_1((1+5^{1/2})/2)^n + \alpha_2((1-5^{1/2})/2)^n.$$

$$f_0 = \alpha_1 + \alpha_2 = 0.$$

$$f_1 = \alpha_1(1+5^{1/2})/2 + \alpha_2(1-5^{1/2})/2 = 1.$$

$$\alpha_1 = 5^{-1/2}, \alpha_2 = -5^{-1/2}.$$

$$\therefore f_n = \alpha_1 r_1^n + \alpha_2 r_2^n = 5^{-1/2}((1+5^{1/2})/2)^n + 5^{-1/2}((1-5^{1/2})/2)^n.$$

Thm. 2 If C.E. of *degree 2* has one **double** roots r_0 , the **solution** is

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n, \text{ for } \forall \alpha_1, \alpha_2 \in \mathbf{R}.$$

proof Since r_0 is a double root. $(r - r_0)^2 = 0$.

Consider derivative, $2r(r - r_0) = 0$.

$\therefore \alpha_2 n r_0^n$ also is a homogeneous solution.

Ex. 5 $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$.

$$\text{C.E } x^2 - 6x + 9 = 0. \quad (x-3)^2 = 0$$

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n. \quad a_0 = \alpha_1 = 1, a_1 = 3\alpha_1 + 3\alpha_2, 3\alpha_2 = 3$$

$$a_n = 3^n + n 3^n.$$

Thm. 3 If C.E. of degree k of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ has k -distinct roots r_1, r_2, \dots, r_k , the **solution** is

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n, \text{ for } \forall \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbf{R}.$$

Ex. 6 C.E. of degree 3 has roots 1, 2, 1/3.

$$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 3^{-n}.$$

Fix α_1, α_2 , and α_3 with 3 initial conditions.

Thm. 4 If C.E. of degree k $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ has t -distinct roots r_1, r_2, \dots, r_t with **implicitness** m_1, m_2, \dots, m_t , where $\sum_t m_t = k$, the **solution** is

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n + \\ (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n + \dots + \\ (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n, \text{ for } \forall \alpha_{i,j} \in \mathbf{R} \ 1 \leq i \leq t, \ 0 \leq j \leq m_i - 1.$$

Ex. 7 Suppose C.E. of degree 6 has roots 2, 2, 2, 5, 5, 9

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)2^n + (\alpha_{2,0} + \alpha_{2,1}n)5^n + \alpha_{3,0}9^n.$$

Fix $\alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,0}, \alpha_{2,1}$, and $\alpha_{3,0}$ with 6 initial conditions.

Linear **nonhomogeneous** recurrence relation of with constant coefficient

Thm. 5 $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$,

Let $a_n^{(h)}$ be a solution the **associated homogeneous** recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}. \text{ Then}$$

Every solution is of the form $a_n = a_n^{(h)} + a_n^{(p)}$ where $a_n^{(p)}$ is called **particular** solution associated with $F(n)$.

Ex. 10 $a_n =_R 3a_{n-1} + 2n$ with $a_1 = 3$.

$$a_n = 3a_{n-1}. \quad a_n^{(h)} = \alpha 3^n.$$

$$F(n) = 2n, \text{ assume } a_n^{(p)} = cn + d. \text{ Then } a_{n-1}^{(p)} =_R 3(c(n-1) + d) + 2n$$

$$\therefore (2+2c)n + (2d - 3c) = 0 \quad \therefore c = -1, d = -3/2.$$

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha 3^n - n - 3/2, \text{ Fix } \alpha \text{ with } a_1 = 3. \alpha = 11/6.$$

$$a_n = (11/6)3^n - n - 3/2.$$

Ex. Tower of Hanoi $H_n = 2H_{n-1} + 1$ with $H_1 = 1$ (degree 1).

associated characteristic equation $r - 2 = 0, r = 2. \therefore H_n^{(h)} = \alpha 2^n.$

$$H_n = H_n^{(p)} + \alpha 2^n.$$

$$F(n) = 1, \text{ try } H_n^{(p)} = C, \quad C = 2C + 1(\text{recurrence}) \quad \therefore C = -1.$$

$$H_1 = -1 + 2\alpha = 1(\text{basis}) \quad \therefore \alpha = 1.$$

$$\therefore H_n = 2^n - 1.$$

Check base case, $H_1 = 2 - 1 = 1$. O.K.

Check the recurrence, $H_n = 2H_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$. O.K.

Ex. 11 $a_n = 5a_{n-1} - 6a_{n-2} + 7^n. \quad a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n.$

$$F(n) = 7^n, \text{ try } a_n^{(p)} = C7^n \quad C7^n = 5C7^{n-1} - 6C7^{n-2} + 7^n.$$

$$7^2 C = 5 \cdot 7 \cdot C - 6C + 49 \quad \therefore C = 49/20.$$

Thm. 6 $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$,

where $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$.

When s is **not** the root of the associated characteristics equation

$$a_n^{(p)} = (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n.$$

When s is the root of the associated C. E. of **multiplicity** m

$$a_n^{(p)} = n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n.$$

Ex. 12 $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$.

root 3 of multiplicity 2

$$F(n) = 3^n \quad a_n^{(p)} = p_0 n^2 3^n.$$

$$F(n) = n 3^n \quad a_n^{(p)} = n^2 (p_1 n + p_0) 3^n.$$

$$F(n) = n^2 2^n \quad a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 2^n.$$

$$F(n) = (n^2 + 1) 3^n \quad a_n^{(p)} = n^2 (p_2 n^2 + p_1 n + p_0) 3^n.$$

Ex. 13 $a_n = a_{n-1} + n$ with $a_1 = 1$.

asso. C. E. $r = 1$ $a_n^{(h)} = \alpha \cdot 1^n = \alpha$.

$$F(n) = n(1)^n, a_n^{(p)} = n(p_1n + p_0) = p_1n^2 + p_0n$$

$$p_1n^2 + p_0n = p_1(n-1)^2 + p_0(n-1) \quad p_0 = p_1 = 1/2.$$

$$a_n^{(p)} = 1/2n^2 + 1/2n = n(n+1)/2$$

$$a_n = n(n+1)/2 + \alpha \text{ with } a_1 = 1.$$

$$a_1 = 1 \cdot 2/2 + \alpha = 1 \quad \therefore \alpha = 0.$$

$$a_n = n(n+1)/2.$$

Basis and recurrence are already checked!

8.3 Divide-and-Conquer Algorithms and Recurrence Relations

Divide-and-Conquer Recurrence Relation

$$f(n) = af(n/b) + g(n) \quad a \geq 1, b \geq 2.$$

Ex. 1 Binary search

$$f(n) = f(n/2) + 2$$

Ex. 3 Merge Sort.

$$M(n) = 2M(n/2) + n$$

$$f(n) = af(n/b) + g(n)$$

$$= a^2f(n/b^2) + ag(n/b) + g(n)$$

$$= a^3f(n/b^3) + a^2g(n/b^2) + ag(n/b) + g(n)$$

...

$$= a^k f(n/b^k) + \sum_{j=0}^{k-1} a^j g(n/b^j)$$

$$\text{When } n = b^k, f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j g(b^{k-j})$$

Thm. 1 $f(n) = af(n/b) + c$ with $a \geq 1$, integer $b \geq 2$, real $c > 0$.

$$f(n) \in O(n^{\log_b a}), \text{ if } a > 1.$$

$$\in O(\log_b n), \text{ if } a = 1.$$

When $n = b^k$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2 \quad \text{where } C_1 = f(1) + c/(a-1) \text{ and } C_2 = -c/(a-1)$$

proof Let $n = b^k$ and $g(n) = c$.

$$f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j c = a^k f(1) + c \sum_{j=0}^{k-1} a^j.$$

$$\text{If } a = 1, f(n) = f(1) + ck = f(1) + c \log_b n \quad \therefore f(n) \in O(\log_b n).$$

$$\begin{aligned} \text{If } a > 1, f(n) &= a^k f(1) + c(a^k - 1)/(a - 1) = a^k [f(1) + c/(a - 1)] - c/(a - 1) \\ &= C_1 a^{\log_b n} + C_2 = C_1 n^{\log_b a} + C_2. \quad \therefore f(n) \in O(n^{\log_b a}) \end{aligned}$$

Ex. 7 Binary search

$$f(n) = f(n/2) + 2$$

$$f(n) \in O(\log_2 n)$$

Thm. 2 Master Theorem

Let $f(n) = af(n/b) + cn^d$ with $a \geq 1$, integer $b > 1$, $c > 0$, $d \geq 0$.

$$f(n) \in O(n^d) \quad \text{if } a < b^d,$$

$$\in O(n^d \log_b n) \quad \text{if } a = b^d,$$

$$\in O(n^{\log_b a}) \quad \text{if } a > b^d.$$

When $n = b^k$, $f(n) = a^k f(1) + c \sum_{j=0}^{k-1} a^j (b^{k-j})^d$.

$$= a^k f(1) + a^{k-1} b^d + a^{k-2} b^{2d} \dots + a^1 b^{(k-1)d} + a^1 b^d.$$

Ex. 9 Merge Sort. $M(n) = 2M(n/2) + n$

$$M(n) \in O(n \log_2 n)$$

8.4 Generating Functions

Def. 1 The generating function for the infinite sequence

$$(a_0, a_1, a_2, a_3, \dots) \leftrightarrow G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$(1, 1, 1, 1, \dots) \leftrightarrow 1 + x + x^2 + x^3 + \dots = 1/(1-x)$$

$$(1, -1, 1, -1, \dots) \leftrightarrow 1 - x + x^2 - x^3 + \dots = 1/(1+x)$$

$$(1, a, a^2, a^3, \dots) \leftrightarrow 1 + ax + a^2x^2 + a^3x^3 + \dots = 1/(1-ax)$$

Rule 1 Scaling Let $(f_0, f_1, f_2, \dots) \leftrightarrow F(x)$. Then

$$(cf_0, cf_1, cf_2, \dots) \leftrightarrow cF(x).$$

proof $(cf_0, cf_1, cf_2, \dots) \leftrightarrow cf_0 + cf_1x + cf_2x^2 + \dots$
 $= c(f_0 + f_1x + f_2x^2 + \dots) = cF(x).$

Rule 2 Addition Let $(f_0, f_1, f_2, \dots) \leftrightarrow F(x)$ and $(g_0, g_1, g_2, \dots) \leftrightarrow G(x)$.

Then $(f_0 + g_0, f_1 + g_1, f_2 + g_2, \dots) \leftrightarrow F(x) + G(x)$.

$$\begin{aligned} \text{proof } (f_0 + g_0, f_1 + g_1, f_2 + g_2, \dots) &\leftrightarrow (f_0 + g_0) + (f_1 + g_1)x + (f_2 + g_2)x^2 + \dots \\ &= f_0 + f_1x + f_2x^2 + \dots + g_0 + g_1x + g_2x^2 + \dots = F(x) + G(x) \end{aligned}$$

Rule 3 Shifting Right Let $(f_0, f_1, f_2, \dots) \leftrightarrow F(x)$. Then

$$(0^1, 0^2, \dots, 0^k, f_0, f_1, f_2, \dots) \leftrightarrow x^k F(x).$$

$$\begin{aligned} \text{proof } (0^1, 0^2, \dots, 0^k, f_0, f_1, f_2, \dots) &\leftrightarrow f_0x^k + f_1x^{k+1} + f_2x^{k+2} + \dots \\ &= x^k(f_0 + f_1x + f_2x^2 + \dots) = x^k F(x). \end{aligned}$$

Rule 4 Differentiation Let $(f_0, f_1, f_2, \dots) \leftrightarrow F(x)$. Then

$$(f_1, 2f_2, 3f_3, \dots) \leftrightarrow d/dx F(x).$$

proof $(f_1, 2f_2, 3f_3, \dots) \leftrightarrow f_1 + 2f_2x^1 + 3f_3x^2 + \dots$
 $= d/dx (f_0 + f_1x + f_2x^2 + \dots) = d/dx (F(x)).$

$$(1, 1, 1, 1, \dots, 1, \dots) \leftrightarrow 1/(1-x).$$

$$(1, 2, 3, 4, \dots, 1+x, \dots) \leftrightarrow d/dx 1/(1-x) = (1-x)^{-2}.$$

$$(0, 1, 2, 3, \dots, x, \dots) \leftrightarrow x \cdot (1-x)^{-2} = x(1-x)^{-2}.$$

$$(1, 4, 9, 16, \dots, (1+x)^2, \dots) \leftrightarrow d/dx x(1-x)^{-2} = (1+x)(1-x)^{-3}.$$

$$(0, 1, 4, 9, \dots, x^2, \dots) \leftrightarrow x(1+x)(1-x)^{-3}.$$

Rule 5 Products Let $(f_0, f_1, f_2, \dots) \leftrightarrow F(x)$ and $(g_0, g_1, g_2, \dots) \leftrightarrow G(x)$.

Then $(f_0g_0, f_0g_1+f_1g_0, f_0g_2+f_1g_1+f_2g_0, \dots) \leftrightarrow F(x)G(x)$.

proof $(f_0g_0, f_0g_1+f_1g_0, f_0g_2+f_1g_1+f_2g_0, f_0g_3+f_1g_2+f_2g_1+f_3g_0, \dots)$

$$\begin{aligned} &\leftrightarrow (f_0g_0) + (f_0g_1+f_1g_0)x + (f_0g_2+f_1g_1+f_2g_0)x^2 + \\ &\quad (f_0g_3+f_1g_2+f_2g_1+f_3g_0)x^3 + (f_0g_4+f_1g_3+f_2g_2+f_3g_1+f_4g_0) + \dots \\ &= F(x)G(x). \end{aligned}$$

Rule 6 Shifting Left/Right Let $(f_0, f_1, f_2, \dots) \leftrightarrow F(x)$. Then

$(0^1, 0^2, \dots, 0^j, f_k, f_{k+1}, f_{k+2}, \dots) \leftrightarrow x^{k-j}F(x) - f_1x^{k-j+1}f_0x^{k-j} - \dots - f_{k-1}x^{2k-j-1}$.

proof $(0^1, 0^2, \dots, 0^j, f_k, f_{k+1}, f_{k+2}, \dots) \leftrightarrow f_kx^j + f_{k+1}x^{j+1} + f_{k+2}x^{j+2} + \dots$

$$\begin{aligned} &= x^{k-j}(f_0 + f_1x + f_2x^2 + \dots + f_{k-1}x^{k-1} + f_kx^k + f_{k+1}x^{k+1} + \dots) \\ &\quad - (f_0x^{k-j} + f_1x^{k-j+1} + f_2x^{k-j+2} + \dots + f_{k-1}x^{2k-j-1}) \\ &= x^{k-j}F(x) - f_1x^{k-j+1}f_0x^{k-j} - \dots - f_{k-1}x^{2k-j-1}. \end{aligned}$$

Fibonacci Sequence

$$f_n = f_{n-1} + f_{n-2} \text{ with } f_0 = 0 \text{ and } f_1 = 1.$$

$$(0, 1, 0, 0, 0, \dots) \leftrightarrow x$$

$$(0, 0, f_1, f_2, f_3, \dots) \leftrightarrow xF(x) - f_0x = xF(x)$$

$$(0, 0, f_0, f_1, f_2, \dots) \leftrightarrow x^2F(x)$$

$$(0, 1, f_1+f_0, f_2+f_1, f_3+f_2, \dots) \leftrightarrow x + xF(x) + x^2F(x)$$

$$F(x) = x + xF(x) + x^2F(x)$$

$$\therefore F(x) = x / (1 - x - x^2)$$

$$= \alpha_1 / (1 - r_1x) + \alpha_2 / (1 - r_2x)$$

$$= \alpha_1(1 + r_1x + r_1^2x^2 + \dots) + \alpha_2(1 + r_2x + r_2^2x^2 + \dots)$$

$$\leftrightarrow \alpha_1(1, r_1, r_1^2, \dots) + \alpha_2(1, r_2, r_2^2, \dots)$$

$$\therefore f_n = \alpha_1 r_1^n + \alpha_2 r_2^n.$$

R.R. of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$ with a_0, a_1, \dots, a_{k-1} .

$$(a_0, a_1, \dots, a_{k-1}, 0, 0, \dots) \leftrightarrow a_0 + a_1 x + \dots + a_{k-1} x^{k-1}.$$

$$c_1(0, 0, \dots, 0, a_{k-1}, a_k, \dots) \leftrightarrow c_1 x [A(x) - a_0 - a_1 x - \dots - a_{k-2} x^{k-1}]$$

$$c_2(0, 0, \dots, 0, a_{k-2}, a_{k-1}, a_k, \dots) \leftrightarrow c_2 x^2 [A(x) - a_0 - a_1 x - \dots - a_{k-3} x^{k-1}]$$

...

$$c_{k-1}(0, 0, \dots, 0, a_1, a_2, \dots, a_{k-1}, a_k, \dots) \leftrightarrow c_{k-1} x^{k-1} [A(x) - a_0]$$

$$c_k(0, 0, \dots, 0, a_0, a_1, \dots, a_{k-1}, a_k, \dots) \leftrightarrow c_k x^k A(x)$$

$$(f_0, f_1, \dots, f_{k-1}, f_k, f_{k+1}, \dots) \leftrightarrow F(x)$$

$$A(x) = [c_1 x + c_2 x^2 + \dots + c_k x^k] A(x) + a_0 + (a_1 - c_1 a_0)x + \\ (a_2 - c_1 a_1 - c_2 a_0)x^2 + \dots + (a_{k-1} - c_1 a_{k-1} - c_2 a_{k-2} \dots - c_{k-1} a_0)x^{k-1} + F(x).$$

$$A(x) = [a_0 + (a_1 - c_1 a_0)x + (a_2 - c_1 a_1 - c_2 a_0)x^2 \dots + (a_{k-1} - c_1 a_{k-1} \dots - c_{k-1} a_0)x^{k-1} \\ + F(x)] / [1 - c_1 x - c_2 x^2 - \dots - c_k x^k]$$

$$\begin{aligned}
A(x) = & \alpha_{1,1}/(1 - r_1x) + \alpha_{1,2}/(1 - r_1x)^2 + \dots + \alpha_{1,m_1}/(1 - r_1x)^{m_1} + \\
& \alpha_{2,1}/(1 - r_2x) + \alpha_{2,2}/(1 - r_2x)^2 + \dots + \alpha_{2,m_2}/(1 - r_2x)^{m_2} + \dots + \\
& \alpha_{t,1}/(1 - r_tx) + \alpha_{t,2}/(1 - r_tx)^2 + \dots + \alpha_{t,m_t}/(1 - r_tx)^{m_t}, \\
& \text{for } \forall \alpha_{i,j} \in \mathbf{R} \ 1 \leq i \leq t, \ 1 \leq j \leq m_i - 1 \text{ where } \sum_t m_t = k.
\end{aligned}$$

Consider r_i of multiplicity m_i $\alpha_{2,i}/(1 - r_ix)^{m_i}$.

$$\begin{aligned}
a_n = & (\alpha_{1,1} + \alpha_{1,2}n + \dots + \alpha_{1,m_1}n^{m_1-1})r_1^n + \\
& (\alpha_{2,1} + \alpha_{2,2}n + \dots + \alpha_{2,m_2}n^{m_2-1})r_2^n + \dots + \\
& (\alpha_{t,1} + \alpha_{t,2}n + \dots + \alpha_{t,m_t}n^{m_t-1})r_t^n, \text{ for } \forall \alpha_{i,j} \in \mathbf{R} \ 1 \leq i \leq t, \ 0 \leq j \leq m_i - 1.
\end{aligned}$$

Binomial Coefficient and Generating Function

$$(\binom{k}{0}, \binom{k}{1}, \binom{k}{2}, \dots, \binom{k}{k}, 0, 0, \dots) \leftrightarrow (\binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 \dots + \binom{k}{k}x^k) \\ = (1 + x)^k$$

Assume the number ways for selecting n -items from set A is a_n .

$$A \leftrightarrow (a_0, a_1, a_2, a_3, \dots) \leftrightarrow A(x)$$

$$\text{Example } \{a\} \leftrightarrow (1, 1, 0, 0, \dots) \leftrightarrow 1 + x$$

$$\{1\} \leftrightarrow (1, 1, 0, 0, \dots) \leftrightarrow 1 + x$$

$$\{a, 1\} \leftrightarrow (1, 2, 1, 0, 0, \dots) \leftrightarrow (1 + x)(1 + x) = 1 + 2x + x^2$$

Rule(Convolution) 6 Let $A \leftrightarrow A(x)$, $B \leftrightarrow B(x)$. Then $A \cup B \leftrightarrow A(x) \cdot B(x)$.

Select n items from $A \cup B$. Select j items from A , and $n-j$ items from B .

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0$$

Choosing Items with Repetition

$$\{a\} \quad (1, 1, 1, \dots) \leftrightarrow 1/(1-x)$$

$$\{a, 1\} \quad ? \quad \leftrightarrow 1/(1-x)^2$$

$\therefore H_k(x) = (1-x)^{-k}$. Is the coefficient of x^n ${}_{n+k-1}C_k$?

Taylor's Expansion

$$f(x) = f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + \dots + f^{(n)}(0)x^n/n! + \dots$$

$$H_k'(x) = k(1-x)^{-(k+1)}.$$

$$H_k''(x) = k(k+1)(1-x)^{-(k+1)}.$$

...

$$H_k^{(n)}(x) = k(k+1)\dots(k+n-1)(1-x)^{-(k+n)}.$$

$$H_k^{(n)}(0)/n! = k(k+1)\dots(k+n-1)/n! = {}_{n+k-1}C_k.$$

8.5 Inclusion-Exclusion

Principle of Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Thm. 1 (The Principle of Inclusion-Exclusion)

$$|A_1 \cup A_2 \cup \dots \cup A_n| =$$

$$\begin{aligned} & \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \dots \\ & + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|. \end{aligned}$$

proof Consider $1 \leq r \leq n$: $a \in A_r$ is counted **exactly once**.

$a \in A_r$ is counted $C(r, 1)$ times in $\sum |A_i|$.

$a \in A_r$ is counted $C(r, 2)$ times in $\sum |A_i \cap A_j|$.

...

$a \in A_r$ is counted $C(r, m)$ times in $\sum |A_i \cap A_j \cap \dots \cap A_m|$ ($m - A_k$ sets)

$\therefore a \in A_r$ is counted $C(r, 1) - C(r, 2) + C(r, 3) - \dots + (-1)^{r+1} C(r, r)$ times.

Since $\sum_{k=0}^n (-1)^k C(n, k) = 0$ by **Col. 2 of Sec. 6.4**(p. 405).

$$\therefore C(r, 1) - C(r, 2) + C(r, 3) - \dots + (-1)^{r+1} C(r, r) = C(r, 0) = 1$$

$\therefore 1 \leq r \leq n: a \in A_r$ is counted **exactly once**.

8.6 Application of Inclusion-Exclusion