8 Advanced Counting Techniques

8.1 Application of Recurrence Relations

***Def**. 1 A **recurrence relation** for a sequence $\{a_n\}$ is*

an equation expresses a_n in terms of $a_0, a_1, \ldots, a_{n-1}$ $\forall n \geq n_0$.

*A sequence is said to be a **solution** of a recurrence, if it consistent with the definition of the recurrence.*

***Ex**. 1 $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$ with $a_0 = 3$ and $a_1 = 5$.*

Ex**. 3 **Compound Interests

$P_n = P_{n-1} + 0.11P_{n-1} = 1.11P_{n-1}$ for $n \geq 1$ with $P_0 = 1$.

$P_n = 1.11P_{n-1} = 1.11(1.11P_{n-2}) = 1.11^3P_{n-3} = \ldots = 1.11^nP_0$.

Ex**. 4 **Rabbits and Fibonacci number

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with $f_0 = 0$ and $f_1 = 1$.

Ex. 5 Tower of Hanoi

1. move (n-1) disks 2. move bottom 1 disk 3. move (n-1) disks.

$H_n = 2H_{n-1} + 1 \; n \geq 2$ with $H_1 = 1$.

$H_n = 2H_{n-1} + 1$

$= 2(2H_{n-2} + 1) = 2^2H_{n-2} + 2 + 1$

$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3H_{n-3} + 2^2 + 2 + 1$

…

$= 2^{n-k-1}H_{k+1} + 2^{n-k-2} + 2^{n-k-3} + \ldots + 2^2 + 2 + 1$

$= 2^{n-k-1}(2H_k + 1) + 2^{n-k-2} + 2^{n-k-3} + \ldots + 2^2 + 2 + 1$

$= 2^{n-k}H_k + 2^{n-k-1} + 2^{n-k-2} + 2^{n-k-3} + \ldots + 2^2 + 2 + 1$

…

$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1$(with $k = 1$)

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1 = 2^n - 1.$

8.2 Solving Linear Recurrence Relations

Def. *1 A* ***linear homogeneous recurrence relation*** *of* ***degree*** *k with* ***constant coefficient*** *is the recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \ \forall n \geq k \text{ with } a_0, a_1, \ldots, a_{k-1}.$$

where $c_1, c_2, \ldots, c_k \in \mathbf{R}$, $c_k \neq 0$.

The ***solution*** *is uniquely determined if k* ***initial conditions*** $a_0, a_1, \ldots, a_{k-1}$ *are provided.*

Characteristic equation

Assume $a_n = r^n$.

$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$, *i.e.,*

$r^k - c_1 r^{k-1} - \ldots c_{k-1} r - c_k = 0$

k ***characteristic roots***

Consider a recurrence relation of degree 2

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

Characteristic equation

$$r^2 - c_1 r - c_2 = 0$$

***Thm**. 1 If C.E. of degree 2 has two **distinct** roots* $r_1 \neq r_2$, ***solution** is*

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n, \text{ for } \forall \alpha_1, \alpha_2 \in \mathbf{R}.$$

***proof** Assume* $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ *is a solution.*

We know $r_1^2 = c_1 r_1 + c_2$ *and* $r_2^2 = c_1 r_2 + c_2$.

$$c_1 a_{n-1} + c_2 a_{n-2} = c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$$

$$= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$$

$$= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$= a_n.$$

Ex. *3.5 small Tower of Hanoi* $h_n = 2h_{n-1}$ *with* $h_1 = 1$*(degree 1).*

characteristic equation $r - 2 = 0,\ r = 2.$

$h_n = \alpha 2^n.$ $h_1 = 2\alpha = 1$ $\therefore \alpha = 2^{-1}.$

$\therefore h_n = 2^{n-1}.$ *Note that* $H_n = 2^n - 1.$

Ex. *4 Fibonacci number* $f_n = f_{n-1} + f_{n-2}$ *with* $f_0 = 0,\ f_1 = 1.$

characteristic equation $r^2 - r - 1 = 0.$

$r_1 = (1+5^{1/2})/2,\ r_2 = (1-5^{1/2})/2.$

$f_n = \alpha_1((1+5^{1/2})/2)^n + \alpha_2((1-5^{1/2})/2)^n.$

$f_0 = \alpha_1 + \alpha_2 = 0.$

$f_1 = \alpha_1(1+5^{1/2})/2 + \alpha_2(1-5^{1/2})/2 = 1.$

$\alpha_1 = 5^{-1/2},\ \alpha_2 = -5^{-1/2}.$

$\therefore f_n = \alpha_1 r_1^n + \alpha_2 r_2^n = 5^{-1/2}((1+5^{1/2})/2)^n + 5^{-1/2}((1-5^{1/2})/2).$

Thm. *2 If C.E. of degree 2 has one* ***double*** *roots r_0, the* ***solution*** *is*

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n, \text{ for } \forall \alpha_1, \alpha_2 \in \mathbf{R}.$$

proof *Since r_0 is a double root. $(r - r_0)^2 = 0$.*

Consider derivative, $2r(r - r_0) = 0$.

$\therefore$ *$\alpha_2 n r_0^n$ also is a homogeneous solution.*

Ex. *5 $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$.*

C.E $x^2 - 6x + 9 = 0$. $(x-3)^2 = 0$

$a_n = \alpha_1 3^n + \alpha_2 n 3^n$. $a_0 = \alpha_1 = 1$, $a_1 = 3\alpha_1 + 3\alpha_2$, $3\alpha_2 = 3$

$a_n = 3^n + n3^n$.

Thm. *3 If C.E. of degree k of* $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ *has k-distinct roots* $r_1, r_2, \ldots r_k$, *the* ***solution*** *is*

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n, \text{ for } \forall \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbf{R}.$$

Ex. *6 C.E. of degree 3 has roots 1, 2, 1/3.*

$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 3^{-n}$.

Fix α_1, α_2, *and* α_3 *with 3 initial conditions.*

Thm. *4 If C.E. of degree k* $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ *has t-distinct roots* $r_1, r_2, \dots r_t$ *with* ***implicitness*** $m_1, m_2, \dots m_t$, *where* $\Sigma_t m_t = k$, *the* ***solution*** *is*

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n + \dots + (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n, \text{ for } \forall\alpha_{i,j} \in \mathbf{R}\ 1 \le i \le t,\ 0 \le j \le m_i-1.$$

Ex. *7 Suppose C.E. of degree 6 has roots 2, 2, 2, 5, 5, 9*

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)2^n + (\alpha_{2,0} + \alpha_{2,1}n)5^n + \alpha_{3,0}9^n.$$

Fix $\alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,0}, \alpha_{2,1}$, *and* $\alpha_{3,0}$ *with 6 initial conditions.*

Linear **nonhomogeneous** recurrence relation of with constant coefficient

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$,

Let $a_n^{(h)}$ be a solution the **associated homogeneous** recurrence relation

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$. Then

Every solution is of the form $a_n = a_n^{(h)} + a_n^{(p)}$ where $a_n^{(p)}$ is called **particular** solution associated with $F(n)$.

Ex. $10\ a_n = 3a_{n-1} + 2n$ with $a_1 = 3$. $a_n^{(h)} = \alpha 3^n$.

$F(n) = 2n$, try $a_n^{(p)} = cn + d$.

$cn + d \equiv 3(c(n-1) + d) + 2n$. check recurrence

$(2c+2)n + (2d - 3c) \equiv 0$ $\therefore\ c = -1, d\ -3/2$.

$a_n = \alpha 3^n - n - 3/2,\ a_1 = 3$ $\alpha = 11/6$ check basis

$a_n = (11/6)3^n - n - 3/2$.

Ex. Tower of Hanoi $H_n = 2H_{n-1} + 1$ with $H_1 = 1$(degree 1).

associated characteristic equation $r - 2 = 0, r = 2. \therefore H_n^{(h)} = \alpha 2^n.$

$H_n = H_n^{(p)} + \alpha 2^n.$

$F(n) = 1$, try $H_n^{(p)} = C$, $C = 2C + 1$(recurrence) $\therefore C = -1.$

$H_1 = -1 + 2\alpha = 1$(basis) $\therefore \alpha = 1.$

$\therefore H_n = 2^n - 1.$

Check base case, $H_1 = 2 - 1 = 1$. O.K.

Check the recurrence, $H_n = 2H_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$. O.K.

Ex. 11 $a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$ $a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n.$

$F(n) = 7^n$, try $a_n^{(p)} = C7^n$ $C7^n = 5C7^{n-1} - 6C7^{n-2} + 7^n.$

$7^2C = 5 \cdot 7 \cdot C - 6C + 49$ $\therefore C = 49/20.$

Thm. *6* $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$,

where $F(n) = (b_tn^t + b_{t-1}n^{t-1} + \ldots + b_1n + b_0)s^n$.

*When s is **not** the root of the associated characteristics equation*

$$a_n^{(p)} = (p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)s^n.$$

*When s is the root of the associated C. E. of **multiplicity** m*

$$a_n^{(p)} = n^m(p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)s^n.$$

Ex. *12* $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$.

root 3 of multiplicity 2

$F(n) = 3^n$ $\quad a_n^{(p)} = p_0n^23^n$.

$F(n) = n3^n$ $\quad a_n^{(p)} = n^2(p_1n + p_0)3^n$.

$F(n) = n^22^n$ $\quad a_n^{(p)} = (p_2n^2 + p_1n + p_0)2^n$.

$F(n) = (n^2 + 1)3^n$ $\quad a_n^{(p)} = n^2(p_2n^2 + p_1n + p_0)3^n$.

Ex. *13* $a_n = a_{n-1} + n$ *with* $a_1 = 1$.

asso. C. E. $r = 1$ $a_n^{(h)} = \alpha \cdot 1^n = \alpha$.

$F(n) = n(1)^n$, $a_n^{(p)} = n(p_1 n + p_0) = p_1 n^2 + p_0 n$

$p_1 n^2 + p_0 n = p_1 (n-1)^2 + p_0 (n-1)$ $p_0 = p_1 = 1/2$.

$a_n^{(p)} = 1/2n^2 + 1/2n = n(n+1)/2$

$a_n = n(n+1)/2 + \alpha$ *with* $a_1 = 1$.

$a_1 = 1 \cdot 2/2 + \alpha = 1$ $\therefore \alpha = 0$.

$a_n = n(n+1)/2$.

Basis and recurrence are already checked!

8.3 Divide-and-Conquer Algorithms and Recurrence Relations

Divide-and-Conquer Recurrence Relation

$$f(n) = af(n/b) + g(n) \qquad a \geq 1, b \geq 2.$$

Ex. 1 Binary search

$$f(n) = f(n/2) + 2$$

Ex. 3 Merge Sort.

$$M(n) = 2M(n/2) + n$$

$$f(n) = af(n/b) + g(n)$$
$$= a^2 f(n/b^2) + ag(n/b) + g(n)$$
$$= a^3 f(n/b^3) + a^2 g(n/b^2) + ag(n/b) + g(n)$$
$$\ldots$$
$$= a^k f(n/b^k) + \sum_{j=0}^{k-1} a^j g(n/b^j)$$

When $n = b^k$, $f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j g(b^{k-j})$

Thm. *1* $f(n) = af(n/b) + c$ *with* $a \geq 1$, *integer* $b \geq 2$, $c > 0$.

$f(n) \in O(n^{\log_b a})$, *if* $a > 1$.

$\in O(\log_b n)$, *if* $a = 1$.

When $n = b^k$, *where k is a positive integer,*

$f(n) = C_1 n^{\log_b a} + C_2$ *where* $C_1 = f(1) + c/(a-1)$ *and* $C_2 = -c/(a-1)$

proof *Let* $n = b^k$ *and* $g(n) = c$.

$f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j c = a^k f(1) + c\sum_{j=0}^{k-1} a^j$.

If $a = 1$, $f(n) = f(1) + ck = f(1) + c\log_b n$ $\therefore O(\log_b n)$.

If $a > 1$, $f(n) = a^k f(1) + c(a^k-1)/(a-1) = a^k[f(1) + c/(a-1)] - c/(a-1)$

$= C_1 a^{\log_b n} + C_2 = C_1 n^{\log_b a} + C_2$. $\therefore O(n^{\log_b a})$

Thm. *2* ***Master Theorem***

Let $f(n) = af(n/b) + cn^d$ *with* $a \geq 1$, *integer* $b > 1$, $c > 0$, $d \geq 0$.

$$
\begin{aligned}
f(n) &\in O(n^d) && \text{if } a < b^d, \\
&\in O(n^d \log_b n) && \text{if } a = b^d, \\
&\in O(n^{\log_b a}) && \text{if } a > b^d.
\end{aligned}
$$

When $n = b^k$, $f(n) = a^k f(1) + c\sum_{j=0}^{k-1} a^j (b^{k-j})^d$.

$$= a^k f(1) + a^{k-1}b^d + a^{k-2}b^{2d} \ldots + a^1 b^{(k-1)d} + a^1 b^d.$$

8.4 Generating Functions

Def. *1 The* ***generating function*** *for the* ***infinite sequence***

$(a_0, a_1, a_2, a_3, \ldots) \leftrightarrow A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$$(1, 1, 1, 1, \ldots) \leftrightarrow 1 + x + x^2 + x^3 + \ldots = 1/(1-x)$$

$$(1, -1, 1, -1, \ldots) \leftrightarrow 1 - x + x^2 - x^3 + \ldots = 1/(1+x)$$

$$(1, a, a^2, a^3, \ldots) \leftrightarrow 1 + ax + a^2x^2 + a^3x^3 + \ldots = 1/(1-ax)$$

Rule *1* ***Scaling*** *Let* $(f_0, f_1, f_2, \ldots) \leftrightarrow F(x)$. *Then*

$$(cf_0, cf_1, cf_2, \ldots) \leftrightarrow cF(x).$$

proof $(cf_0, cf_1, cf_2, \ldots) \leftrightarrow cf_0 + cf_1x + cf_2x^2 + \ldots$

$$= c(f_0 + f_1x + f_2x^2 + \ldots) = cF(x).$$

Rule *2* ***Addition*** *Let* $(f_0, f_1, f_2, \ldots) \leftrightarrow F(x)$ *and* $(g_0, g_1, g_2, \ldots) \leftrightarrow G(x)$.
Then $(f_0 + g_0, f_1 + g_1, f_2 + g_2, \ldots) \leftrightarrow F(x) + G(x)$.

proof $(f_0 + g_0, f_1 + g_1, f_2 + g_2, \ldots) \leftrightarrow (f_0 + g_0) + (f_1 + g_1)x + (f_2 + g_2)x^2 \ldots$

$$= f_0 + f_1x + f_2x^2 + \ldots g_0 + g_1x + g_2x^2 + \ldots = F(x) + G(x)$$

Rule *3* ***Shifting Right*** *Let* $(f_0, f_1, f_2, \ldots) \leftrightarrow F(x)$. *Then*

$$(0^1, 0^2, \ldots, 0^k, f_0, f_1, f_2, \ldots) \leftrightarrow x^kF(x).$$

proof $(0^1, 0^2, \ldots, 0^k, f_0, f_1, f_2, \ldots) \leftrightarrow f_0x^k + f_1x^{k+1} + f_2x^{k+2} + \ldots$

$$= x^k(f_0 + f_1x + f_2x^2 + \ldots) = x^kF(x).$$

Rule** 4 **Differentiation *Let* $(f_0, f_1, f_2, \ldots) \leftrightarrow F(x)$. *Then*

$$(f_1, 2f_2, 3f_3, \ldots) \leftrightarrow d/dx\ F(x).$$

proof $(f_1, 2f_2, 3f_3, \ldots) \leftrightarrow f_1 + 2f_2x^1 + 3f_3x^2 + \ldots$

$$= d/dx\ (f_0 + f_1x + f_2x^2 + \ldots) = d/dx\ (F(x).$$

$(1, 1, 1, 1, \ldots, 1, \ldots) \leftrightarrow 1/(1\text{-}x).$

$(1, 2, 3, 4, \ldots, 1+x, \ldots) \leftrightarrow d/dx\ 1/(1\text{-}x) = (1\text{-}x)^{-2}.$

$(0, 1, 2, 3, \ldots, x, \ldots) \leftrightarrow x \cdot (1\text{-}x)^{-2} = x(1\text{-}x)^{-2}.$

$(1, 4, 9, 16, \ldots, (1+x)^2, \ldots) \leftrightarrow d/dx\ x(1\text{-}x)^{-2} = (1+x)(1\text{-}x)^{-3}.$

$(0, 1, 4, 9, \ldots, x^2, \ldots) \leftrightarrow x(1+x)(1\text{-}x)^{-3}.$

Rule *5* ***Products*** *Let* $(f_0, f_1, f_2, \ldots) \leftrightarrow F(x)$ *and* $(g_0, g_1, g_2, \ldots) \leftrightarrow G(x)$.

Then $(f_0g_0, f_0g_1+f_1g_0, f_0g_2+f_1g_1+f_2g_0, \ldots) \leftrightarrow F(x)G(x)$.

proof $(f_0g_0, f_0g_1+f_1g_0, f_0g_2+f_1g_1+f_2g_0, f_0g_3+f_1g_2+f_2g_1+f_3g_0, \ldots)$

$\leftrightarrow (f_0g_0) + (f_0g_1+f_1g_0)x + (f_0g_2+f_1g_1+f_2g_0)x^2 +$

$(f_0g_3+f_1g_2+f_2g_1+f_3g_0)x^3 + (f_0g_4+f_1g_3+f_2g_2+f_3g_1+f_4g_0) + \ldots$

$= F(x)G(x)$.

Rule *6* ***Shifting Left/Right*** *Let* $(f_0, f_1, f_2, \ldots) \leftrightarrow F(x)$*. Then*

$$(0^1, 0^2, \ldots, 0^j, f_k, f_{k+1}, f_{k+2}, \ldots) \leftrightarrow x^{k-j}F(x) - f_1x^{k-j+1}f_0x^{k-j} - \ldots - f_{k-1}x^{2k-j-1}.$$

proof $(0^1, 0^2, \ldots, 0^j, f_k, f_{k+1}, f_{k+2}, \ldots) \leftrightarrow f_kx^j + f_{k+1}x^{j+1} + f_{k+2}x^{j+2} + \ldots$

$$= x^{k-j}(f_0 + f_1x + f_2x^2 + \ldots + f_{k-1}x^{k-1} + f_kx^k + f_{k+1}x^{k+1} + \ldots)$$

$$- (f_0x^{k-j} + f_1x^{k-j+1} + f_2x^{k-j+2} + \ldots + f_{k-1}x^{2k-j-1})$$

$$= x^{k-j}F(x) - f_1x^{k-j+1}f_0x^{k-j} - \ldots - f_{k-1}x^{2k-j-1}.$$

Fibonacci Sequence

$f_n = f_{n-1} + f_{n-2}$ *with* $f_0 = 0$ *and* $f_1 = 1$.

$(0, 1, 0, 0, 0, \ldots) \leftrightarrow x$

$(0, 0, f_1, f_2, f_3, \ldots) \leftrightarrow xF(x) - f_0x = xF(x)$

$(0, 0, f_0, f_1, f_2, \ldots) \leftrightarrow x^2F(x)$

$(0, 1, f_1+f_0, f_2+f_1, f_3+f_2, \ldots) \leftrightarrow x + xF(x) + x^2F(x)$

$F(x) = x + xF(x) + x^2F(x)$

$\therefore F(x) = x / (1 - x - x^2)$

$= \alpha_1 / (1 - r_1x) + \alpha_2 / (1 - r_2x)$

$= \alpha_1(1 + r_1x + r_1^2x^2 + \ldots) + \alpha_2(1 + r_2x + r_2^2x^2 + \ldots)$

$\leftrightarrow \alpha_1(1, r_1, r_1^2, \ldots) + \alpha_2(1, r_2, r_2^2, \ldots)$

$\therefore f_n = \alpha_1r_1^n + \alpha_2r_2^n$.

R.R. of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n)$ with $a_0, a_1, \ldots, a_{k-1}$.

$(a_0, a_1, \ldots, a_{k-1}, 0, 0, \ldots) \leftrightarrow a_0 + a_1 x + \ldots + a_{k-1} x^{k-1}$.

$c_1(0, 0, \ldots, 0, a_{k-1}, a_k, \ldots) \leftrightarrow c_1 x[A(x) - a_0 - a_1 x - \ldots - a_{k-2} x^{k-1}]$

$c_2(0, 0, \ldots, 0, a_{k-2}, a_{k-1}, a_k, \ldots) \leftrightarrow c_2 x^2[A(x) - a_0 - a_1 x - \ldots - a_{k-3} x^{k-1}]$

…

$c_{k-1}(0, 0, \ldots, 0, a_1, a_2 \ldots, a_{k-1}, a_k, \ldots) \leftrightarrow c_{k-1} x^{k-1}[A(x) - a_0]$

$c_k(0, 0, \ldots, 0, a_0, a_1, \ldots, a_{k-1}, a_k, \ldots) \leftrightarrow c_k x^k A(x)$

$(f_0, f_1, \ldots, f_{k-1}, f_k, f_{k+1}, \ldots) \leftrightarrow F(x)$

$A(x) = [c_1 x + c_2 x^2 + \ldots + c_k x^k]A(x) + a_0 + (a_1 - c_1 a_0)x +$

$(a_2 - c_1 a_1 - c_2 a_0)x^2 + \ldots + (a_{k-1} - c_1 a_{k-1} - c_2 a_{k-2} \ldots - c_{k-1} a_0)x^{k-1} + F(x)$.

$A(x) = [a_0 + (a_1 - c_1 a_0)x + (a_2 - c_1 a_1 - c_2 a_0)x^2 \ldots + (a_{k-1} - c_1 a_{k-1} \ldots - c_{k-1} a_0)x^{k-1}$

$+F(x)] \;/\; [1 - c_1 x - c_2 x^2 - \ldots - c_k x^k]$

$A(x) = \alpha_{1,1}/(1-r_1x) + \alpha_{1,2}/(1-r_1x)^2 + \ldots + \alpha_{1,m_1}/(1-r_1x)^{m_1} +$
$\alpha_{2,1}/(1-r_2x) + \alpha_{2,2}/(1-r_2x)^2 + \ldots + \alpha_{2,m_2}/(1-r_2x)^{m_2} + \ldots +$
$\alpha_{t,1}/(1-r_tx) + \alpha_{t,2}/(1-r_tx)^2 + \ldots + \alpha_{t,m_t}/(1-r_tx)^{m_t},$
$\text{for } \forall\alpha_{i,j} \in \mathbf{R}\ 1\le i\le t,\ 1\le j\le m_i-1 \text{ where } \Sigma_t\, m_t = k.$

Consider r_i *of multiplicity* m_i $\alpha_{2,i}/(1-r_ix)^{m_i}$.

$a_n = (\alpha_{1,1} + \alpha_{1,2}n + \ldots + \alpha_{1,m_1}n^{m_1-1})r_1^{\,n} +$
$(\alpha_{2,1} + \alpha_{2,2}n + \ldots + \alpha_{2,m_2}n^{m_2-1})r_2^{\,n} + \ldots +$
$(\alpha_{t,1} + \alpha_{t,2}n + \ldots + \alpha_{t,m_t}n^{m_t-1})r_t^{\,n}, \text{ for } \forall\alpha_{i,j} \in \mathbf{R}\ 1\le i\le t,\ 0\le j\le m_i-1.$

Binomial Coefficient and Generating Function

$({}_kC_0, {}_kC_1, {}_kC_2, \ldots, {}_kC_k, 0, 0, \ldots) \leftrightarrow ({}_kC_0 + {}_kC_1x + {}_kC_2x^2 \ldots + {}_kC_kx^k$
$= (1 + x)^k$

Assume the number ways for selecting n-items from set A is a_n.

$A \leftrightarrow (a_0, a_1, a_2, a_3, \ldots) \leftrightarrow A(x)$

Example $\{a\} \leftrightarrow (1, 1, 0, 0, \ldots) \leftrightarrow 1 + x$

$\{1\} \leftrightarrow (1, 1, 0, 0, \ldots) \leftrightarrow 1 + x$

$\{a, 1\} \leftrightarrow (1, 2, 1, 0, 0, \ldots) \leftrightarrow (1 + x)(1 + x) = 1 + 2x + x^2$

Rule(Convolution) *6 Let* $A \leftrightarrow A(x)$, $B \leftrightarrow B(x)$. *Then* $A \cup B \leftrightarrow A(x)\cdot B(x)$.
Select n items from $A \cup B$. *Select j items from A, and n-j items from B.*

$a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \ldots + a_nb_0$

Choosing Items with Repetition

$\{a\}$ $(1, 1, 1, \ldots) \leftrightarrow 1/(1 - x)$

$\{a, 1\}$? $\leftrightarrow 1/(1 - x)^2$

$\therefore H_k(x) = (1 - x)^{-k}$. Is the coefficient of x^n ${}_{n+k-1}C_k$?

Taylor's Expansion

$$f(x) = f(0) + f'(0)x + f''(0)x^2 + f'''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n! + \ldots$$

$H_k'(x) = k(1-x)^{-(k+1)}$.

$H_k''(x) = k(k+1)(1-x)^{-(k+1)}$.

…

$H_k^{(n)}(x) = k(k+1)\ldots(k+n-1)(1-x)^{-(k+n)}$.

$H_k^{(n)}(0)/n! = k(k+1)\ldots(k+n-1)/n! = {}_{n+k-1}C_k$.

8.5 Inclusion-Exclusion

8.6 Application of Inclusion-Exclusion