

7 Discrete Probability

7.1 An Introduction to Discrete Probability

Experiment a procedure that yields one of the given set of **outcomes**

Sample space of an experiment universe of possible **outcomes**

Event a subset of the sample space

Definition 1 If S is a **finite** sample space of equally likely outcomes and $E (\subseteq S)$ is an **event**, the **probability** of E is, $p(E) = |E|/|S|$. We say the event E **occurs** when **actual outcome** e is in E , $e \in E$.

Theorem 1 Let E be an event in a sample space S . The probability of the event \bar{E} , the **complementary** event of E , is

$$p(\bar{E}) = 1 - p(E).$$

Theorem 2 Let E_1 and E_2 be events in a sample space S . Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

7.2 Probability Theory

Let S be **countable** sample space and $s \in S$ be an outcome, $p(s)$ be the probability of the outcome s .

$$i) \forall s \in S, 0 \leq p(s) \leq 1.$$

$$ii) \sum_{s \in S} p(s) = 1.$$

Probability distribution The function $p: S \rightarrow [0, 1]$

Definition 1 Let S be a set with n elements. The **uniform(random) distribution** assigns the probability $1/n$ to each element of S .

Definition 2 Let E be an event. Then

$$p(E) = \sum_{s \in E} p(s).$$

Theorem 1 If E_1, E_2, \dots is a sequence of **pairwise disjoint** events,

$$p(\cup_i E_i) = \sum_i p(E_i).$$

Definition 3 Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F is

$$p(E | F) = p(E \cap F) / p(F).$$

Definition 4 The event E and F **independent** if and only if

$$p(E \cap F) = p(E)p(F).$$

Bernoulli Trial and the Binomial distribution

experiments with two possible outcomes **{success, failure}**

$$p + q = 1$$

Theorem 2 The probability of k successes in n **independent** Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$C(n, k) p^k q^{n-k}.$$

Random Variables

Definition 5 Let S be a sample space S and A be a set of values of random variable. **Random variable** X is a function from set of sample space(events) to some set of value A .

$$X: S \rightarrow A.$$

Definition 6 The **distribution** of random variable X is a pair $(s, p(X=s))$

$\forall s \in S, p(X=s)$ is the probability that X takes the value s .

We write $p(X=s)$ instead of pair $(s, p(X=s))$.

7.3 Bayes' Theorem

Theorem 1 Baye's Theorem Suppose E and F are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$.

$$\begin{aligned} p(F | E) &= p(E | F)p(F) / (p(E | F)p(F) + p(E | \bar{F})p(\bar{F})) \\ &= p(E | F)p(F) / p(E) \end{aligned}$$

$$p(F | E)p(E) = p(E | F)p(F).$$

proof

$$p(F | E) = p(E \cap F) / p(E).$$

$$p(E | F) = p(E \cap F) / p(F).$$

$$p(F | E)p(E) = p(E | F)p(F) = p(E \cap F).$$

$$p(E) = p(E | F)p(F) + p(E | \bar{F})p(\bar{F})$$

7.4 Expected Value and Variance

Definition 1 The *expected value* (or *expectation*) of the random variable $X(s)$ on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s)X(s).$$

Theorem 1 The *expected value* of the random variable $p(X=s)$, then

$$E(X) = \sum_{s \in S} p(X=s)s.$$

Theorem 2 The *expected number of successes* when n independent Bernoulli trials are performed, where p is the probability of success of each trial, is np .

$$\begin{aligned} \text{proof } E(X) &= \sum_{k=1}^n p(X=k)k = \sum_{k=1}^n kC(n, k)p^k q^{n-k} \\ &= \sum_{k=1}^n nC(n-1, k-1)p^k q^{n-k} = np \sum_{k=1}^n C(n-1, k-1)p^{k-1} q^{n-k} \\ &= np \sum_{j=0}^{n-1} C(n-1, j)p^j q^{n-1-j} = np(p+q)^{n-1} = np \end{aligned}$$

Theorem 3 Let X_1, X_2, \dots, X_n are random variables on S , and $a, b \in \mathbf{R}$.

$$(i) E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$$

$$(ii) E(aX + b) = aE(X) + b.$$

Average-Case Computational Complexity

Let S be the sample space of an algorithm and $a \in S$ be input data.

$$E(X) = \sum_{a \in S} p(a)X(a).$$

Geometric Distribution

Definition 2 A random variable X has a **geometric distribution** with parameter p , if $p(X=k) = (1 - p)^{k-1}p$ for $k \in \mathbf{N}$.

Theorem 4 If the random variable X has the geometric distribution with parameter p , then $E(X) = 1/p$.

proof See Example 10

Independent Random Variable

Definition 3 A random variable X and Y on a space S are **independent**, if
$$p(X(s) = r_1 \text{ and } Y(s) = r_2) = p(X(s) = r_1)p(Y(s) = r_2).$$

Theorem 5 If X and Y are **independent** random variables on a space S ,
$$E(XY) = E(X)E(Y).$$

Definition 4 Let X be a random variable on a space S . The **variance** of X ,
 $V(X)$, is
$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

The **standard deviation** of X , $\sigma(X) = (V(X))^{1/2}$.

Theorem 6 If X is a random variables on a space S , then

$$V(X) = E(X^2) - E(X)^2.$$

Theorem 7 If X and Y are **independent** random variables on a space S ,
$$V(X + Y) = V(X) + V(Y).$$