6 Counting

6.1 The Basics of Counting

<u>Product and Sum Rules</u>

Suppose two task,

m number of ways to do task 1

n number of ways to do task 2

*Do **both** task 1 **and** task 2:* **Product Rule**

mn ways

$|A \times B| = |A| \cdot |B|$.

*Do **either** task 1 **or** task 2, but not both:* **Sum Rule**

m+n ways

$|A \cup B| = |A| + |B|$, if $A \cap B = \emptyset$.

<u>Principle of Inclusion-Exclusion</u>

$|A \cup B| = |A| + |B| - |A \cap B|$.

6.2 The Pigeonhole Principle

Theorem** 1 **The pigeonhole principle

If k + 1 or more objects are placed into k box,

there is at least one or more box containing two or more objects.

Dirichlet drawer principle

***Corollary** 1 If f: $A \rightarrow B$ and $|A| \geq |B|+1$, then f is **not** one-to-one.*

*If f: $A \rightarrow B$ and $|A| > |B|$, then f is **not** one-to-one.*

Theorem** 2 **The generalized pigeonhole principle

If N objects are placed into k box,

there is at least one or more box containing $\lceil N/k \rceil$ objects.

***proof** by **contradiction**, no box contains $\lceil N/k \rceil$ objects.*

$k(\lceil N/k \rceil - 1) < k((N/k+1) - 1) = N$

***Example** N = 59 students, k = 12 months,*

$\lceil N/k \rceil = \lceil 4.91... \rceil = 5$ *students/month*

6.3 Permutations and Combinations

***Thm**. 1* For $n \geq 0$ and $1 \leq \forall r \leq n$, $P(n, r) = n(n-1) \ldots (n-r-1)$

***Fact**. 1* $P(n, 0) = n$.

***Col**. 1* For $n \geq 0$ and $0 \leq \forall r \leq n$, $P(n, r) = n! / (n-r)!$

***Thm**. 2* $0 \leq r \leq n$, $C(n, r) = P(n, r) / P(r, r) = n! / r!(n-r)!$

***Col**. 2* $C(n, r) = C(n, n\text{-}r)$

6.4 Binomial Coefficient

Thm. 1 $(x+y)^n = \sum_{j=0}^{n} C(n, j) \cdot x^{n-j} \cdot y^j$

$= C(n, 0) \cdot x^n \cdot y^0 + C(n, 1) \cdot x^{n-1} \cdot y^1 + \ldots + C(n, n-k) \cdot x^{n-k} \cdot y^k + \ldots +$
$C(n, n-1) \cdot x^1 \cdot y^{n-1} + C(n, n) \cdot x^0 \cdot y^n.$ *for* $1 \leq \forall k \leq n$.

proof $C(n, n-k)$ *terms of* $x^{n-k} \cdot y^k$ *among* 2^n *expaned terms of* $(x+y)^n$.

Col. 1 $\sum_{k=0}^{n} C(n, k) = 2^n$ *for* n.

proof $2^n = (1+1)^n = \sum_{k=0}^{n} C(n, k) 1^k 1^{n-k} = \sum_{k=0}^{n} C(n, k)$.

Col. 2 $\sum_{k=0}^{n} (-1)^k C(n, k) = 0$.

$0 = 0^n = (1+(-1))^n = \sum_{k=0}^{n} C(n, k)(-1)^k 1^{n-k} = \sum_{k=0}^{n} (-1)^k C(n, k)$.

Col. 3 $\sum_{k=0}^{n} 2^k C(n, k) = 3^n$.

$3^n = (1+2)^n = \sum_{k=0}^{n} C(n, k) 1^{n-k} 2^k = \sum_{k=0}^{n} C(n, k) 2^k$.

Pascal's Identity and Trialgle

Thm. *2* $C(n+1, k) = C(n, k-1) + C(n, k)$

Proof $C(n, k-1) + C(n, k) = n! / [(n-k+1)!\cdot(k-1)!] + n! / [(n-k)!\cdot k!]$

$= [n!\cdot k + n!\cdot(n-k+1)] / [(n-k+1)!\cdot k!]$

$= n!(n+1) / [(n-k+1)!\cdot k!] = (n+1)! / [(n-k+1)!\cdot k!] = C(n+1, k)$

Pascal's triangle

$C(0, 0)$

$C(1, 0)\ C(1, 1)$

…

$C(n, 0) \ldots C(n, k-1)\ C(n, k) \ldots C(n, n)$

$C(n+1, 0) \ldots C(n+1, k-1)\ C(n+1, k)\ C(n+1, k+1) \ldots C(n, n)$

6.5 Generalized Permutation and Combinations

Permutation with Repetition

Thm 1 $\Pi(n, r) = n \cdot n \cdot \ldots \cdot n = n^r$.

$P(n, r) = n! / (n-r)!$

Combination with Reprtition

Thm 2 $H(n, r) = C(n + r - 1, r) = C(n + r - 1. n - 1)$

$= (n+r - 1)!/r!(n-1)!$

$C(n, r) = n!/ r!(n-r)!$

6.6 Generating Permutation and Combinations

Let $a_1a_2\ldots a_n$ *and* $b_1b_2\ldots b_n$ *be* ***permutation*** *of* $\{1, 2, \ldots, n\}$.

We say $a_1a_2\ldots a_n < b_1b_2\ldots b_n$,

if $1 \le \exists k \le n$.э. $a_1 = b_1, a_2 = b_2, \ldots, a_{k-1} = b_{k-1}, a_k < b_k$.

permutation $a_1a_2\ldots a_n \in \{1, \ldots, n\}^n$.

$<$ *is a total order*

Next permutation of $a_1 \ldots a_n$?

Find largest j .϶. $a_j < a_{j+1}$.

Find smallest interger a_k in $a_{j+1}, \ldots, a_n$ but $a_k > a_j$.

Interchange a_k and a_j.

Increasing order from j+1-th to n-th position.

procedure *nextPermutation*($a_1 \ldots a_n \in \{1, \ldots, n\}^n \neq n\ (n-1) \ldots 2\ 1$)

$j := n-1$; **do** $a_j > a_{j+1} \rightarrow j := j - 1$ **od**

$k := n$; **do** $a_j > a_k \rightarrow k := k - 1$ **od**

$a_k, a_j := a_j, a_k$;

$r, s := n, j+1$; **do** $r > s \rightarrow a_r, a_s := a_s, a_r$; $r, s := r - 1, s+1$ **od**

procedure *next bit string*($b_1 \ldots b_n \in \{0, 1\}^n \neq 111 \ldots 11$)

$i := n$; **do** $b_i = 1 \rightarrow b_i := 0; i := i - 1$ **od**; $b_i := 1$.

Next r-combination of $\{a_1, a_2, \ldots, a_n\}$ *with* $a_1 < a_2 < \ldots < a_n$?

Find a_i .϶. $a_i \neq n - r + i$.

replace a_i *with* $a_i + 1$.

repalace a_j*'s with* $a_i + j - i$.

procedure *next r-comnination*($\{a_1, a_2, \ldots, a_n\} \subseteq \{1, 2, \ldots, n\}$

$\neq \{n\text{-}r\text{+}1, n\text{-}r\text{+}2, \ldots, n\}$ *and* $a_1 < a_2 < \ldots < a_n$)

$i := r$; **do** $a_i = n - r + i \rightarrow i := i - 1$ **od**; $a_i := a_i + 1$;

for $j{:=}i{+}1$ ***to*** r ***do*** $a_j := a_i + j - i$ ***od***