1 The Foundation: Logic and Proofs

1.1 Propositional Logic

Propositions(명제)

A declarative ***sentence*** *that is either* ***true*** *or* ***false****,*
but ***not both nor neither****.*

propositional *(or* ***statement****;* 문 *)* ***variables***(변수)

p, q, r, s, …

propositional constants(상수)

T: ***truth*** *value(****true****)*

F: ***falsehood*** *value(****false****)*

propositional calculus, ***propositional logic*** *or* ***FOL(First Order Logic)***

Greek philosopher Aristotle *2300 years ago*

Compounded propositions(복합명제)

propositions *that are* ***formed*** *from* ***existing*** *propositions*
using ***logical operators(connectives****;* $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$*)*

Def*. 1* ***Negation(Not****;* $\neg$*)*
Let p be a proposition, $\neg p$ *is a (new)* ***compounded*** *proposition,*
called ***negation*** *of p, or "****not*** *p".*

Def*. 2* ***Conjunction(And****;* $\wedge$*)*
Let p and q be a propositions, $p \wedge q$ *is a (compounded) proposition,*
called ***conjunction*** *of p and q, or "p* ***and*** *q".*

Def*. 3* ***Disjunction(Or, Inclusive or****;* $\vee$*)*
Let p and q be a propositions, $p \vee q$ *is a proposition,*
called ***disjunction*** *of p and q, or "p* ***or*** *q".*

$\neg$ ***unary prefix operator***

$\wedge, \vee, \oplus, \rightarrow, \leftrightarrow$ ***binary infix operator***

Def. 4 Exclusive or *(⊕)* *See Rosen's TP p. 11*

Let p and q be a propositions, p ⊕ q is a proposition,
called ***exclusive or*** *of p and q, or "p* ***xor*** *q".*

Def. *5* ***Conditional (Implication*** 조건 *; →)*

Let p and q be a propositions, p → q is a proposition,
called ***implication*** *of p and q, or "p implies q".*
"if p, then q", "p, only if q", …(See p. 15 of Rosen's TP)
p is called ***hypothesis***(가정), ***premise*** *or* ***antecedent*** (전건)*of p → q*
q is called ***conclusion***(결론) *or* ***consequence***(후건) *of p → q*

See truth table of ***implication****(Table 5)*

p → q is ***false*** *only in the case that p is* ***true*** *an q is* ***false****.*
logic says ***nothing*** *when the* ***hypothesis*** *is* ***false***
p → q is true when p is false (inclusive) or q is true
∴ p → q is ***equivalent*** *to ¬p ∨ q.*

Let $p \to q$ be a implication proposition. Then

*$q \to p$ is a **converse** (역) of $p \to q$,*
*$\neg q \to \neg p$ is a **contrapositive** (대우) of $p \to q$, and*
*$\neg p \to \neg q$ is a **inverse** (이) of $p \to q$.*

*$p \to q$ and **contrapositive** $\neg q \to \neg p$ are **equivalent**.*
***converse** $q \to p$ and **inverse** $\neg p \to \neg q$ are **equivalent**.*

Def**. 6 **biconditional (equivalence)

Let p and q be a propositions, $p \leftrightarrow q$ is a proposition,
*called **biconditional** of p and q, or "p, if and only if, q".*

$p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$.
$p \leftrightarrow q$ is equivalent to $\neg(p \oplus q)$.

Truth Table of Compound Proposition

1 proposition	*2 rows in the truth table*
2 propositions	*4 rows in the truth table*
…	
n propositions	2^n *rows in the truth table*

Precedence of Logical Operators

$\neg$	*highest*
$\wedge$	
$\vee$	…
$\rightarrow$	
$\leftrightarrow$	*lowest*

$p \vee q \wedge \neg r \rightarrow s = (p \vee (q \wedge (\neg r))) \rightarrow s$

Logic and Bit Operations

T	*1*
F	*0*

Def. *7 A* ***bit string*** *is a* ***sequence*** *of zero or more bits.*
The ***length*** *of a string is the number of bits in the string.*

1.2 Application of Logic Programs

Translating English Sentences

System Specifications

Boolean Searches

Logic Puzzle

Exa. 7 ***knight(truth teller)*** *or* ***knave(liar)***

A says "B is a ***knight****."*

B says "Two of us are ***opposite*** *types."*

???

Logic *or* ***Digital Circuits***

Three logic gates *(Fig. 1)*

1. inverter or NOT gate(¬) 2. ∨: OR gate(∨) 3. AND gate(∧)

Normal Forms(Sec. ***1.7****)*

Combinatorial circuit*(Fig. 2)*

1.3 Propositional Equivalencies

Def. *1 A (compound) proposition that is always true(***T***):* ***tautology***

*A proposition that is always false(***F***):* ***contradiction***

*Otherwise(***T** *or* **F***):* ***contingency***(우연성).

Ex. 1 $p \vee \neg p \equiv \mathbf{T}$ *and* $p \wedge \neg p \equiv \mathbf{F}$.

Logical Equivalences

Def. *2 Two propositions p and q are* ***logically equivalent****, if* $p \leftrightarrow q$ *is a* ***tautology****, written,* $p \equiv q$ *or* $p \Leftrightarrow q$.

Ex. 2 $\neg(p \vee q) \equiv \neg p \wedge \neg q$ ***De Morgan's law***

Ex. 3 $p \rightarrow q \equiv \neg p \vee q$ ***normal form***$(\neg \wedge \vee)$ *elimination of* $\rightarrow$

Ex. 4 $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ ***Disitibutive law***

truth table

Algebraic rules of logical equivalencies *(Table 6, p. 25)*

1. ***Identity laws***	$p \vee \mathbf{F} \equiv p$	$p \wedge \mathbf{T} \equiv p$
2. ***Domination laws***	$p \vee \mathbf{T} \equiv \mathbf{T}$	$p \wedge \mathbf{F} \equiv \mathbf{F}$
3. ***Idempotent laws***	$p \vee p \equiv p$	$p \wedge p \equiv p$
4. ***Double negation law***	$\neg(\neg p) \equiv p$	
5. ***Commutative laws***	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
6. ***Associative laws***	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
7. ***Distributive laws***	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8. ***DeMorgan's laws***	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
9. ***Absortion laws***	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
10. ***Negation laws***	$p \vee \neg p \equiv \mathbf{T}$	$p \wedge \neg p \equiv \mathbf{F}$
11. ***T/F laws***	$\neg\mathbf{T} \equiv \mathbf{F}$	$\neg\mathbf{F} \equiv \mathbf{T}$

Logical equivalencies involving conditional statements (Table 7)

$p \to q \equiv \neg p \vee q$ *disjunctive normal form (elimination of $\to$)*

$p \to q \equiv \neg q \to \neg p$ *contrapositive*

$p \vee q \equiv \neg p \to q$ $\quad p \wedge q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q$ *conjunctive normal form*

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ $\quad (p \to q) \wedge (r \to q) \equiv (p \vee r) \to q$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$ $\quad (p \to q) \vee (r \to q) \equiv (p \wedge r) \to q$

Logical equivalencies involving biconditional statements (Table 8)

$p \leftrightarrow q \equiv q \leftrightarrow p$ *symmetricity of biconditional*

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ *definition of biconditional*

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ *symmetricity of biconditional*

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ *disjunctive normal form*

$\equiv (p \vee \neg q) \wedge (\neg p \vee q)$ *conjunctive normal form*

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$ *De Mogran's law for bicond.*

Since disjunction($\vee$) and conjunction($\wedge$) are ***associative****,*

$p \vee q \vee r$ *and* $p \wedge q \wedge r$ *are* ***well defined****.*

Let $p_1, p_2, \ldots, p_n$ *be n propositions. Then* (***binary*** *op.* $\subset$ ***n-ary*** *op.)*

$p_1 \vee p_2 \vee \ldots \vee p_n = \vee_{i=1}^{n} p_j = \vee_{i \in \{1, 2, \ldots, n\}} p_j = \vee_{i \in \mathbf{N}_{1,n}} p_j$ *and*

$p_1 \wedge p_2 \wedge \ldots \wedge p_n = \wedge_{i=1}^{n} p_j = \wedge_{i \in \{1, 2, \ldots, n\}} p_j = \wedge_{i \in \mathbf{N}_{1,n}} p_j$ *are* ***well defined****.*

Extended De Morgan's Laws

$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\quad\quad$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg \vee_{i \in \mathbf{N}_{1,n}} p_j \equiv \wedge_{i \in \mathbf{N}_{1,n}} \neg p_j$ $\quad\quad$ $\neg(\wedge_{i \in \mathbf{N}_{1,n}} \neg p_j) \equiv \vee_{i \in \mathbf{N}_{1,n}} \neg p_j$

Constructing New Logical Equivalences

Two *ways to* ***prove*** *the logical equivalencies*

1. ***truth tables*** $\quad$ *n propositions,* 2^n ***raws*** *in the table*

   Venn diagram $\quad$ *n propositions,* 2^n ***areas*** *in the diagram*

2. ***algebraic rules*** *of logical equivalencies* $\quad$ *Ex. 6, 7, 8*

1.4 Predicates and Quantifiers

Predicate(조건명제): *a* ***proposition*** *with* ***variable***

A predicate P(x) has the proposition P and the variable x

Ex. 1 Let P(x) denotes "x > 3". Then

P(4) denotes "4 > 3" is **T**. *P(2) denote "2 > 3" is* **F**.

Let D be an ***universe(or domain) of discourse (set)*** *for x. Then*

$P: D \rightarrow \{\mathbf{T}, \mathbf{F}\}$ ***propositional(boolean) function***

Ex. Let O(n) denotes "n is an odd number". Then

O(3) is **T** *and P(4)* **F** *are* ***propositions***.

But O(n) is ***not*** *a* ***proposition***.

Predicate is a ***boolean function***, *but* ***not*** *a proposition.*

But! …

Quantifiers

*A **predicate** is **not** a **proposition** only if, **variables** are **not fixed**.*
*If **all** the variables are fixed, the predicate **becomes** a propositions.*
How can we fix variables?

*Consider **universe of discourse**(**domain set**) for each **variable**.*

*If P(x) is true for **all** values of x in the **universe of discourse**(U),*
$\forall xP(x)$ *is true*
otherwise $\forall xP(x)$ *is false.*
$\therefore \forall xP(x)$ *becomes a **proposition**.*

predicate calculus

***Def.** 1 **Universal quantifier**($\forall$)*

$\forall xP(x)$ *is a proposition such that*

"P(x) for ***all*** *values in the* ***domain****."*

$\forall$ *is called* ***universal quantifier****.*

We read $\forall xP(x)$ *as "****for all*** *x P(x)."*

Let a set $D = \{x_1, x_2, \ldots, x_n\}$ *be the* ***domain of discourse*** *for x. Then*

$$\forall x \in D\ P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

An element $c \in D$ *.ə.* $P(c) = \mathbf{F}$ *is called*

a ***counterexample*** *of* $\forall xP(x)$.

function** "*$^{\forall}x \in D\ P(x)$*" ∈ **B ***(Exa. 1.5.7, Johnsonbaugh)***

for $d \in D$ ***do if*** $P(d) \rightarrow$ ***skip*** $| \neg P(d) \rightarrow$ ***return*** **F** ***fi od; return*** **T**.

건너뛰기

***Def.** 2 **Existential quantifier**(∃)*

∃xP(x) is a proposition such that

*"There **exists** an element x in the **domain** such that P(x)."*

*∃ is called **existential quantifier**.*

*We read ∃xP(x) as "**there is at least one** x such that P(x)."*

*Let a set D = {x_1, x_2, …, x_n} be the **domain of discourse** for x. Then*

$$\forall x \in D\ P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n).$$

*An element c ∈ D .ə. P(c) = **T** is a **good example** of ∃xP(x).*

function** "$^{\exists}x \in D\ P(x)$" ∈ **B** **(Exa. 1.5.12, Johnsonbaugh)

***for** d ∈ D **do if** P(d) → **return T** | ¬P(d) → **skip fi od**; **return F**.*

Indexed set notation

Let a set $D = \{x_1, x_2, \ldots, x_n\}$ *be the* ***domain of discourse*** *for x. Then*

$$\forall x \in D\ P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n) \equiv \wedge_{i=1}^{n} P(x_1)$$

$$\equiv \wedge_{i \in \{1, 2, \ldots, n\}} P(x_i) \equiv \wedge_{x \in D} P(x)$$ ***indexed*** *set notation.*

$$\exists x \in D\ P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n) \equiv \vee_{i=1}^{n} P(x_1)$$

$$\equiv \vee_{i \in \{1, 2, \ldots, n\}} P(x_i) \equiv \vee_{x \in D} P(x).$$

Uniqueness quantifier

$\exists! x \in D\ P(x)$ *or* $\exists_1 x \in D\ P(x)$

"There is a ***unique*** *x such that P(x)."*

Binding variables

*A variable is said to be **bound**, if the variable binds to*

(1) quantifiers(∀, ∃) or

(2) specific value(in the domain), and

*it is said to be **free**, otherwise.*

*If **all** of the variables in a predicate is **bound**,*

*then the predicate is a **proposition**.*

*Otherwise [Or there is **one or more free** variables] it is **not**.*

Negating Quantified Expressions

$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$ $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

$\forall$ and $\exists$ has higher precedence than $\neg$, $\wedge$, or $\vee$.

Proof: De Morgan's Law

Let D be a universe of discourse. Then

$\neg\forall x \in D\, P(x) = \neg\wedge_{x\in D} P(x_j) \equiv \vee_{x\in D} \neg P(x_j) = \exists x \neg P(x).$

$\neg\exists x \in D\, P(x) = \neg\vee_{x\in D} P(x_j) \equiv \wedge_{x\in D} \neg P(x_j) = \forall x \neg P(x).$

Translating from English into Logical expressions

Examples from Lewis Carroll

Alice in Wondeland

Ex. 26

Logic Programming

Disjunction of ***predicates****(literals) with* ***at most one un-negated*** *literals.*

$$\neg P_1(X_1) \vee \neg P_2(X_2) \vee \ldots \vee \neg P_n(X_n) \vee P(X)$$
$$\equiv \neg(P_1(X_1) \wedge P_2(X_2) \wedge \ldots \wedge P_n(X_n)) \wedge P(X)$$
$$\equiv (P_1(X_1) \wedge P_2(X_2) \wedge \ldots \wedge P_n(X_n)) \Rightarrow P(X)$$

$$H(X) \Leftarrow B_1(X_1) \wedge B_2(X_2) \wedge \ldots \wedge B_n(X_n)$$

Four *cases for head and body literals and* ***three*** *clauses*

1. ***querry*** *clause* $\Leftarrow B_1(X_1) \wedge B_2(X_2) \wedge \ldots \wedge B_n(X_n)?$
2. ***rule*** *cluases* $H(X) \Leftarrow B_1(X_1), B_2(X_2), \ldots, B_n(X_n).$
3. ***fact*** *clauses* $H(X).$
4. *tautology* .

Logic program(Prolog)

A ***querry*** *clause and* ***rules*** *and* ***facts****.*

database *Ex. 28*

Syntax of Prolog

Clause	***querry*** *clause*	*body* ***LiteralS*** **?**
	rule *clause*	*head Literal* **:–** *body* ***LiteralS*** **.**
	fact *clause*	*head Literal* **.**
Literal		***predicate (termS)***
Term		***constant*** | ***variable*** | *literal*

Ex. ***map coloring***

:– *map(A, B, C, D, E)***?**

map(A, B, C, D, E) **:–** *next(A, B), next(A, C), next(A, D),*
next(B, C), next(B, E), next(C, D), next(C, E), next(D, E).

next(X, Y) **:–** *diff(X, Y)* | *diff(Y, X).*

diff(red, blue).

diff(red, green).
diff(blue, green).

diff(red, yellow).
diff(blue, yellow).
diff(green, yellow).

1.5 Nested Quantifier

Ex. 1 $\forall x \in \mathbf{R} \forall y \in \mathbf{R}\ (x+y = y+x)$ ***commutative law*** *for* $+$

$\forall x \in \mathbf{N} \exists y \in \mathbf{Z}\ (x+y = 0)$ ***extend*** $\mathbf{N}$ *to* $\mathbf{Z}$ *for* $-$

$\forall x \in \mathbf{R} \forall y \in \mathbf{R} \forall z \in \mathbf{R}\ (x+(y+z) = (x+y)+z)$ ***associative law*** *for* $+$

The Order of Quantifiers

$\forall x \forall y\ P(x, y) \equiv \forall y \forall x\ P(x, y)$

function "$\forall x \in D\ \forall y \in E\ P(x, y)$ *or* $\forall y \forall x\ P(x, y)$" $\in \mathbf{B}$ $\forall x \forall y\ P(x, y)$

for $x \in D$ ***do for*** $y \in E$ ***do if*** $P(x, y) \rightarrow$ ***skip*** | $\neg P(x, y) \rightarrow$ ***return F fi od od; return T***

for $y \in E$ ***do for*** $x \in D$ ***do if*** $P(x, y) \rightarrow$ ***skip*** | $\neg P(x, y) \rightarrow$ ***return F fi od od; return T***

for $(x, y) \in D \times E$ ***do if*** $P(x, y) \rightarrow$ ***skip*** | $\neg P(x, y) \rightarrow$ ***return F fi od; return T***

$\exists x \exists y\ P(x, y) \equiv \exists y \exists x\ P(x, y)$

function "$\exists x \in D \exists y \in E\ P(x, y)$ *or* $\exists y \exists x\ P(x, y)$" $\in \mathbf{B}$ $\exists x \exists y\ P(x, y)$

for $(x, y) \in D \times E$ ***do if*** $P(x, y) \rightarrow$ ***return F*** | $\neg P(x, y) \rightarrow$ ***skip fi od; return F***

Ex. 4 $\exists y \in \mathbf{R}\ \forall x \in \mathbf{R}\ (x+y = 0) \equiv \mathbf{F}$ $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R}\ (x+y = 0) \equiv \mathbf{T}$

$\forall x\ \exists y\ P(x, y) \not\equiv \exists y\ \forall x\ P(x, y)$

function "$\forall x \in D\ \exists y \in E\ P(x, y)$" $\in$ ***B*** $\forall x \exists y\ P(x, y)$

for $x \in D$ ***do*** $\forall x$

for $y \in E$ ***do*** $\exists y$

if $P(x, y) \rightarrow$ ***break*** | $\neg P(x, y) \rightarrow$ ***skip fi od;***

if $P(x, y) \rightarrow$ ***skip*** | $y \notin E \rightarrow$ ***return F fi od;***

return T

function "$\exists x \in D\ \forall y \in E\ P(x, y)$" $\in$ ***B*** $\exists x \forall y\ P(x, y)$

for $x \in D$ ***do*** $\exists x$

for $y \in E$ ***do*** $\forall y$

if $P(x, y) \rightarrow$ ***skip*** | $\neg P(x, y) \rightarrow$ ***break fi od;***

if $y \notin E \rightarrow$ ***return T*** | $\neg P(x, y) \rightarrow$ ***skip fi od;***

return F

$\exists y\ \forall x\ P(x, y) \Rightarrow \forall x\ \exists y\ P(x, y)$.

Let $x_1, x_2, \ldots, x_n$ be n variables with ***domain of discourses*** $D_1, D_2, \ldots, D_n$.
Then ***predicate*** $P(x_1, x_2, \ldots, x_n)$ *is*

$$P: (D_1 \times D_2 \times \ldots \times D_n) \rightarrow \{\mathbf{T}, \mathbf{F}\}$$

Ex. 4 Let Q(x, y) denotes "x+y=0"
$\forall x \exists y(x+y=0)$ *vs* $\exists y \forall x(x+y=0)$

Translating Statements involving Nested Quantifiers
Translating Sentences into Logical Expressions
Negating Nested Quantifier

Ex. 12 $\neg \forall x \exists y(xy=1) \equiv \exists x \neg \exists y(xy=1) \equiv \exists x \forall y(\neg xy=1) \equiv \exists x \forall y(xy \neq 1)$.

1.6 Rules of Inference

Proof: ***valid arguments*** *that*
establish the truth of mathematical statements

arguments *a sequence of statement that ends with a* ***conclusion***

valid *the* ***conclusion*** *must follow from the truth of the* ***preceding statements*** *or* ***premises of the argument***

An ***argument*** *is* ***valid****, if and only if,*

it is impossible for ***all premises*** *to be* ***true*** *and* ***conclusion*** *to be* ***false***

or If all premises are true, then the conclusion is true.

Rules of inference

deducing *new statements from statements we already have.*

propositional logic

Incorrect reasoning

fallacies

rules of inference for ***qualified*** *statements*

Valid Arguments in Propositional Logic

Definition ***argument*** *a sequence of propositions preceding* ***premises*** *and finally a* ***conclusion***.

argument form

$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$

valid *argument*

$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ *is* ***tautology***.

$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Rightarrow q$.

p_1

p_2

$\ldots$

p_n

q.

Modus ponen 전건긍정

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore\ q \end{array} \qquad [p \wedge (p \to q)] \Rightarrow q$$

Modus tollen 후건부정

$$\begin{array}{l} \neg q \\ \underline{p \to q} \\ \therefore\ \neg p \end{array} \qquad [\neg q \wedge (p \to q)] \Rightarrow \neg p$$

Hypothetical syllogism

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore\ p \to r \end{array} \qquad [(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$$

Disjunctive syllogism

$$\begin{array}{l} p \vee q \\ \underline{\neg p\ \ \ } \\ \therefore\ q \end{array} \qquad [(p \vee q) \wedge \neg p] \Rightarrow q$$

Addition

$$\begin{array}{l} \underline{p\ \ \ \ } \\ \therefore\ p \vee q \end{array} \qquad p \Rightarrow (p \vee q)$$

Simplification

$$\begin{array}{l} \underline{p \wedge q} \\ \therefore\ p \end{array} \qquad (p \wedge q) \Rightarrow p$$

Conjunction

$$\begin{array}{l} p \\ \underline{q\ \ \ \ } \\ \therefore\ p \wedge q \end{array} \qquad [(p) \wedge (q)] \Rightarrow (p \wedge q)$$

Resolution

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore\ q \vee r \end{array} \qquad [(p \vee q) \wedge (\neg p \vee r)] \Rightarrow (q \vee r)$$

Example 6 Prove that four hypotheses

(H1) "It is not sunny and its cold."	$\neg sunny \wedge cold$
(H2) "We will swim only if it is sunny."	$swim \rightarrow sunny$
(H3) "If we do not swim, then we will canoe."	$\neg swim \rightarrow canoe$
(H4) "If we canoe, then we will be home early,"	$canoe \rightarrow early$

Concludes

"We will be home early"	*early*

proof

1. $\neg sunny \wedge cold$	*Hypothesis(H1)*
2. $\neg sunny$	*Simplification using (1)*
3. $swim \rightarrow sunny$	*Hypothesis(H2)*
4. $\neg swim$	*Modus tollen using (2) and (3)*
5. $\neg swim \rightarrow canoe$	*Hypothesis(H3)*
6. $canoe$	*Modus ponen using (4) and (5)*
7. $canoe \rightarrow early$	*Hypothesis(H4)*
8. $early$	*Q.E.D.*

Resolution

$((p \vee q) \wedge (\neg p \vee r)) \Rightarrow (q \vee r)$

$((p \vee q) \wedge (\neg p \vee q)) \Rightarrow q$

$((p \vee q) \wedge (\neg p)) \Rightarrow q$

Fallacies

$((p \rightarrow q) \wedge q) \not\Rightarrow p$ ***fallacy** of **affirming** the **conclusion***

$((p \rightarrow q) \wedge \neg p) \not\Rightarrow \neg q$ ***fallacy** of **denying** the **hypothesis***

Logic says nothing when hypotheses are false!

Rules of inferences for qualified Statements

Universal instantiation

$\forall x P(x)$

$\therefore P(c)$

Universal generalization

$P(c)$ *for an arbitrary* c

$\therefore \forall x P(x)$

Existential instantiation

$\exists x P(x)$

$\therefore P(c)$ *for some element* c

Existential generalization

$P(c)$ *for some element* c

$\therefore \exists x P(x)$

1.7 Normal Forms

*A **well formed formula**(wff) of propositional logic is a **string** over propositional **constants**(**T**, **F**), propositional **variables**, **connectives** $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and parenthesis in **proper** manner.*

*Syntax **grammar** for (well-formed) propositions*

$p = \mathbf{T} \mid \mathbf{F} \mid v \mid \neg p \mid p \wedge p \mid p \vee p \mid p \rightarrow p \mid p \leftrightarrow p \mid (p)$

***Def**. 0 A **constants**, **variable** or the **negation** of variable is a **literal**.*

$p, q, r, \ldots, \neg p, \neg q, \ldots$

*An **elementary product** is a **product**(conjunction) of **literals**.*

$p \wedge q \wedge \ldots \wedge \neg p \wedge \ldots \equiv \ldots \wedge \mathbf{F} \wedge \ldots \equiv \mathbf{F}$.

*An **elementary sum** is a **sum**(disjunction) of **literals**.*

$p \vee q \vee \ldots \vee \neg q \vee \ldots \equiv \ldots \vee \mathbf{T} \vee \ldots \equiv \mathbf{T}$.

***Def**. 1 A formula which is **equivalent** to a given formula and consists of a **sum** of **elementary product** is called a **disjunctive normal form**(DNF).*

***Def**. 2 A formula which is **equivalent** to a given formula and consists of a **Product** of **elementary sum** is called a **conjunctive normal form**(CNF).*

*DNF and CNF are **not** unique.*

Let $p_1, p_2, \ldots, p_n$ be n propositions in a formula. Then
*a **minterm** of the formula is an elementary **product***
$q_1 \wedge q_2 \wedge \ldots \wedge q_n$ *where* $q_i = p_i$ *or* $q_i = \neg p_i$.
*a **maxterm** of the formula is an elementary **sum***
$q_1 \vee q_2 \vee \ldots \vee q_n$ *where* $q_i = p_i$ *or* $q_i = \neg p_i$.

Consider a binary number $i = b_1 b_2 \ldots b_n$ *with* $0 \le i < 2^n$,

where $b_i = 1$, *if* $q_i = p_i$ *and* $b_i = 0$, *if* $q_i = \neg p_i$.

minterm $m_i = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ *and* ***Maxterm*** $M_i = q_1 \vee q_2 \vee \ldots \vee q_n$.

2^n *minterms and* 2^n *Maxterms.*

Example $m_0 = \neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n$, $M_{2^n-1} = p_1 \vee p_2 \vee \ldots \vee p_n$.

$\neg m_0 \equiv M_{2^n-1}$.

Def*. 3 A formula consisting of a* ***sum*** *of* ***minterms*** *only is called a* ***principal disjunctive normal form****(PDNF).*

sum of ***true*** *raws in the truth table*

Def*. 4 A formula consisting of a* ***product*** *of* ***maxterms*** *only is called a* ***principal conjunctive normal form****(PCNF).*

Let a PCNF of a logical formula be $\vee_{b_i} m_i$. *Then an equivalant PDNF is*

$\neg\neg\vee_{b_i} m_i \equiv \neg\wedge_{b_i} \neg m_i \equiv \neg\wedge_{\overline{b}_i} M_i$. *where* $\overline{b}_i$ *is a* ***complement*** *of* b_i.

The ***negation*** *of product of* ***false*** *raws in the truth table is PCNF.*
PDNF and PCNF are ***unique****.*

Example

p	q	b	*minterm*	*maxterm*	$p\rightarrow q$	$\neg(p\rightarrow q)$
F	***F***	*00*	$\neg p\wedge\neg q$	$\neg p\vee\neg q$	***T***	***F***
F	***T***	*01*	$\neg p\wedge q$	$\neg p\vee q$	***T***	***F***
T	***F***	*10*	$p\wedge\neg q$	$p\vee\neg q$	***F***	***T***
T	***T***	*11*	$p\wedge q$	$p\vee q$	***T***	***F***

$p\rightarrow q \equiv (\neg p\wedge\neg q) \vee (\neg p\wedge q) \vee (p\wedge q) \equiv \overline{p}\overline{q}+\overline{p}q+pq$ *PDNF*

$\equiv \neg(p\vee\neg q) \equiv \neg(p+\overline{q})$ *PCNF*

$\neg(p\rightarrow q) \equiv \neg(p\wedge\neg q) \equiv p\overline{q}$ *PDNF*

$\equiv \neg[(\neg p\vee\neg q) \wedge (\neg p\vee q) \wedge (p\vee q)] \equiv \neg[(\overline{p}+\overline{q})(\overline{p}+q)(p+q)]$ *PCNF*

1.8 Introduction to Proofs

Some Terminologies

***Theorem**: A statement that has been proven to be **true**.*

***Axiom: Assumption** to be true (often unproven)*
*defining the **structures** about which we are reasoning.*

Rules of inference**: **Patterns** of logically valid **deductions
*from **hypotheses to conclusion.***

Lemma**: A **minor** theorem used as a **stepping stone
to prove a major theorem

***Corollary**: A **minor** theorem proven*
*as an **easy consequence** of a major theorem*

***Conjecture**: A statement whose truth value has not been proven.*
(A conjecture may be widely believed to be true, regardless)

***Theory**: The set of **all theorems** that*
*can be proven from a **given** set of **axioms***

Direct Proof

$\forall x(P(x) \rightarrow Q(x))$

$P(c) \rightarrow Q(c)$ *universal generalization* ($\Uparrow$)

$p \rightarrow q$ *propositional calculus*

Example 1 "If n is odd integer, then n^2 *is odd"*

proof $\forall n(O(n) \Rightarrow O(n^2))$ *where O(n) is "n is odd"*

$n = 2k + 1$ $O(n)$

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ $O(n^2)$

$O(n) \Rightarrow O(n^2)$ *implication*

$\forall n(O(n) \Rightarrow O(n^2))$ *universal generalization*

Proof by Contrapositive

$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

Example 3 "If n is an integer and 3n + 2 is odd, then n is odd"

proof $3n + 2 = 2k + 1$, $n = ?$

If n is even, then 3n+2 is even.
n = 2k, 3n+2 = 6k + 2 = 2(3k + 1) is even.

Vacuous proof

*If p = **F**, p ⇒ q is a tautology.*

See Section 4.1 Mathematical induction

Trivial proof

*If q= **T**, p ⇒ q is a tautology.*

See Section 4.1 Mathematical induction

Proofs by Contradiction

*If ¬p ⇒ q, q = **F**, p is a tautology.*
If ¬p ⇒ (r ∧ ¬r), p is a tautology.
*If ¬p ⇒ **F**, p is a tautology.*

Example 9

p = "At least four of any 22 days must fall on the same day of the week"

$\neg p$ = "At most three of 22 days ..."

r = "22 days are chosen"

$\neg p \rightarrow (r \wedge \neg r)$, p is a tautology.

Example 10 Prove that $\sqrt{2}$ is irrational.

p = "$\sqrt{2}$ is irrational" *$\neg p$ = "$\sqrt{2}$ is rational"*

$\sqrt{2}$ = a/b, a and b are integers.

$2 = a^2/b^2$ *$2b^2 = a^2$.*

a^2 is even, a is also even. a = 2c.

$a^2 = 4c^2 = 2b^2$. *$b^2 = 2c^2$.*

b^2 is even. b is even.

*$\sqrt{2}$ = a/b, a and b are **even** integers*

Proof of Equivalence

$(p \Leftrightarrow q) \equiv [(p \Rightarrow q) \wedge (q \Rightarrow p)]$

$(p_1 \Leftrightarrow p_2 \Leftrightarrow \ldots \Leftrightarrow p_n) \equiv [(p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3) \wedge \ldots \wedge (p_n \Rightarrow p_1)]$

Counter Example

To prove $\forall x P(x)$ is false

an example x P(x) is false

Mistakes in Proof

Example 16 divide by zero

Example 17 $(p \Rightarrow q)$ does not implies $(q \Rightarrow p)$

Example 18 $(p \Rightarrow q)$ does not implies $(\neg p \Rightarrow \neg q)$

1.9 Proof Methods and Strategy

Exhaustive Proof and Proof by Case

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \Rightarrow q] \equiv [(p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q) \wedge \ldots \wedge (p_n \Rightarrow q)]$$

Existence Proofs

Uniqueness Proofs

Proof Statuettes

Looking for Counterexamples

Proof Strategy in Action

Tilings

The Role of Open Problems

Additional Proof Methods